On minimizing the norm of linear maps in $C_p$-classes

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Abstract. In this paper we establish various characterizations of the global minimum of the map $F_\psi : U \to \mathbb{R}^+$ defined by $F_\psi(X) = \|\psi(X)\|_{p'}$, $(1 < p < \infty)$ where $\psi : U \to C_p$ is a map defined by $\psi(X) = S + \phi(X)$ and $\phi : B(H) \to B(H)$ is a linear map, $S \in C_p$, and $U = \{X \in B(H) : \phi(X) \in C_p\}$. Further, we apply these results to characterize the operators which are orthogonal to the range of elementary operators.


Key words: elementary operators, Schatten $p$-classes, orthogonality, $\varphi$-directional derivative.

1 Introduction

Let $E$ be a complex Banach space. We recall ([2]) that $b \in E$ is orthogonal to $a \in E$ (in short $b \perp a$) if for all complex $\lambda$ there holds $\|a + \lambda b\| \geq \|a\|$. Note that the order is important, that is, if $b$ is orthogonal to $a$, then $a$ need not be orthogonal to $b$. If $E$ is a Hilbert space, then $b \perp a$ is equivalent to $\langle a, b \rangle = 0$, i.e., the orthogonality in the usual sense. Let now $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space $H$ and let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq ... \geq 0$ denote the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator $T$ is said to be belong to the Schatten $p$-classes $C_p$ if

$$\|T\|_p = \left[ \sum_{i=1}^{\infty} s_i(T)^p \right]^{\frac{1}{p}} = [\text{tr}|T|^p]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

where $\text{tr}$ denotes the trace functional. For the general theory of the Schatten $p$-classes the reader is referred to [11]. Recall (see [11]) that the norm $\|\cdot\|$ of the B-space $V$ is said to be Gâteaux differentiable at non-zero elements $x \in V$ if there exists a unique support functional $D_x \in V^*$ such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$ and satisfying

$$\lim_{R, t \to 0} \frac{\|x + ty\| - \|x\|}{t} = \text{Re} D_x(y).$$
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for all \( y \in V \). Here \( \mathbb{R} \) denotes the set of all reals and \( \Re \) denotes the real part. The Gâteaux differentiability of the norm at \( x \) implies that \( x \) is a smooth point of the sphere of radius \( \| x \| \). It is well known (see [7] and the references therein) that for \( 1 < p < \infty \), \( C_p \) is a uniformly convex Banach space. Therefore every non-zero \( T \in C_p \) is a smooth point and in this case the support functional of \( T \) is given by

\[
DT(X) = \text{tr} \left[ \left( \frac{|T|^{p-1} UX^*}{\|T\|^{p-1}} \right) \right],
\]

for all \( X \in C_p \), where \( T = U |T| \) is the polar decomposition of \( T \). The first result concerning the orthogonality in a Banach space was given by Anderson [1] showing that if \( A \) is a normal operator on a Hilbert space \( H \), then \( AS = SA \) implies that for any bounded linear operator \( X \) there holds

\[
\| S + AX - XA \| \geq \| S \| .
\] (1.1)

This means that the range of the derivation \( \delta_A : B(H) \to B(H) \) defined by \( \delta_A(X) = AX - XA \) is orthogonal to its kernel. This result has been generalized in two directions: by extending the class of elementary mappings

\[
E : B(H) \to B(H); \quad E(X) = \sum_{i=1}^n A_i X B_i
\]

and

\[
\tilde{E} : B(H) \to B(H); \quad \tilde{E}(X) = \sum_{i=1}^n A_i X B_i - X,
\]

where \( (A_1, A_2, \ldots A_n) \) and \( (B_1, B_2, \ldots B_n) \) are \( n \)-tuples of bounded operators on \( H \), and by extending the inequality (1.1) to \( C_p \)-classes with \( 1 < p < \infty \) see [4], [8]. The Gâteaux derivative concept was used in [3, 5, 7, 9, 10], in order to characterize those operators which are orthogonal to the range of a derivation. In these papers, the attention was directed to \( C_p \)-classes for some \( p \geq 1 \). The main purpose of this note is to characterize the global minimum of the map

\[
X \mapsto \| S + \phi(X) \|_{C_p}, \quad \phi \text{ is a linear map in } B(H),
\]

in \( C_p \) by using the \( \varphi \)-directional derivative. These results are then applied to characterize the operators \( S \in C_p \) which are orthogonal to the range of elementary operators. It is very interesting to point out that our Theorem 3.3 and its Corollary 3.2 generalize Theorem 1 in [9] and Lemma 2 in [3].

2 Preliminaries

Definition 2.1 Let \((X, \| \cdot \|)\) be an arbitrary Banach space and \( F : X \to \mathbb{R} \). We define the \( \varphi \)-directional derivative of \( F \) at a point \( x \in X \) in direction \( y \in X \) by

\[
D_\varphi F(x; y) = \lim_{t \to 0^+} \frac{F(x + te^{i\varphi}y) - F(x)}{t}.
\]
Note that when $\varphi = 0$ the $\varphi$-directional derivative of $F$ at $x$ in direction $y$ coincides with the usual directional derivative of $F$ at $x$ in a direction $y$ given by

$$(2.1) \quad DF(x; y) = \lim_{t \to 0^+} \frac{F(x + ty) - F(x)}{t}.$$ 

According to the notation given in [6] we will denote $D_\varphi F(x; y)$ for $F(x) = \|x\|$ by $D_\varphi,x(y)$ and for the same function we write $D_x(y)$ for $DF(x; y)$.

**Remark 2.1** In [6] the author used the term $\varphi$-Gâteaux derivative instead of the term “$\varphi$-directional derivative” that we use here. It seems to us that the most appropriate term is the “$\varphi$-directional derivative”, because in the classical case when we don’t have $\varphi$, as in (2.1) the existence of this limit corresponds to the directional differentiability of $F$ at $x$ in the direction $y$, while the Gâteaux differentiability of $F$ at $x$ corresponds to the existence of the same limit in any direction $y \in E$ and moreover the function $y \mapsto DF(x; y)$ is linear and continuous. We note that the existence of $DF(x; y)$ for any $y \in E$ does not imply the Gâteaux differentiability of $F$ at $x = 0$ does not exist.

We recall (see [8, Proposition 6]) that the function $y \mapsto D_{\varphi,x}(y)$ is subadditive and

$$(2.2) \quad |D_{\varphi,x}(y)| \leq \|y\|.$$ 

We end this section by recalling a necessary optimality condition in terms of $\varphi$-directional derivative for a minimization problem.

**Theorem 2.1** ([10]) Let $(X, \|\cdot\|)$ be an arbitrary Banach space and $F : X \to \mathbb{R}$. If $F$ has a global minimum at $v \in X$, then

$$\inf_{\varphi} D_{\varphi} F(v; y) \geq 0,$$

for all $y \in X$.

**3 Main Results**

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha \phi(X) + \beta \phi(Y)$, for all $\alpha, \beta \in \mathbb{C}$ and all $X, Y \in B(H)$, and let $S \in C_p (1 < p < \infty)$. Put

$$U = \{X \in B(H) : \phi(X) \in C_p\}.$$ 

Let $\psi : U \to C_p$ be defined by

$$\psi(X) = S + \phi(X).$$

Define the function $F_\psi : U \to \mathbb{R}^+$ by $F_\psi(X) = \|\psi(X)\|_{C_p}$. Now we are ready to prove our first result in $C_p$-classes ($1 < p < \infty$). It gives a necessary and sufficient optimality condition for minimizing $F_\psi$. The proof of this result follows, with slight modifications, the same lines of the proof of Theorem 3.1 in [10]. For the convenience of the reader we state it.
Theorem 3.1 The map $F_\psi$ has a global minimum at $V \in U$ if and only if
\[
\inf_{\phi} D_{\phi,\psi}(\phi(Y)) \geq 0, \ \forall \ Y \in U.
\]

Proof. For the necessity we have just to combine Theorem 2.1 and the following equality which can be easily checked
\[
D_{\phi,\psi}(V, Y) = D_{\phi,\psi}(V)(\phi(Y)).
\]

Conversely, assume that (3.1) is satisfied. First, observe that
\[
D_{\phi,\psi}(V)(e^{i(\pi-\varphi)}\psi(V)) = \lim_{t \to 0^+} \left\| \psi(V) + te^{i\varphi}e^{i(\pi-\varphi)}\psi(V) \right\|_{C_p} - \left\| \psi(V) \right\|_{C_p} \cdot
\]
\[
= \lim_{t \to 0^+} \left( \frac{\left\| \psi(V) - t\psi(V) \right\|_{C_p} - \left\| \psi(V) \right\|_{C_p}}{t} \right) \cdot \frac{|1 - t| - 1}{t} = -\left\| \psi(V) \right\|_{C_p}.
\]

From this, we have
\[
\left\| \psi(V) \right\|_{C_p} = -D_{\phi,\psi}(V)(e^{i(\pi-\varphi)}\psi(V)).
\]

Let $Y \in U$ be arbitrary and put $\widetilde{Y} = -e^{i(\pi-\varphi)}Y + e^{i(\pi-\varphi)}V$. It is easy to see that $\widetilde{Y} \in U$. Then by (3.1) we have $D_{\phi,\psi}(V)(\phi(\widetilde{Y})) \geq 0$ and hence by the subadditivity of $D_{\phi,\psi}(V)(.)$ and the linearity of $\phi$ we get
\[
\left\| \psi(V) \right\|_{C_p} \leq -D_{\phi,\psi}(V)(e^{i(\pi-\varphi)}\psi(V)) + D_{\phi,\psi}(V)(\phi(\widetilde{Y}))
\]
\[
= D_{\phi,\psi}(V)(-e^{i(\pi-\varphi)}\phi(Y) + e^{i(\pi-\varphi)}\phi(V) - e^{i(\pi-\varphi)}S - e^{i(\pi-\varphi)}\phi(V))
\]
\[
= D_{\phi,\psi}(V)(-e^{i(\pi-\varphi)}\psi(Y)).
\]

By using (2.2) and since $Y$ is arbitrary in $U$, we obtain
\[
F_\psi(V) = \left\| \psi(V) \right\|_{C_p} \leq D_{\phi,\psi}(V)(-e^{i(\pi-\varphi)}\psi(Y)) \leq \left\| \psi(Y) \right\|_{C_p} = F_\psi(Y), \text{ for all } Y \in U.
\]

Then $F_\psi$ has a global minimum at $V$ on $U$. \hfill $\Box$

Let us recall the following result proved in [9] for $C_p$-classes ($1 < p < \infty$).

Theorem 3.2 ([9]) Let $X, Y \in C_p$. Then, there holds
\[
D_X(Y) = p\text{Re} \left\{ \text{tr}(|X|^{p-1}U^*Y) \right\},
\]
where $X = U |X|$ is the polar decomposition of $X$.

The following corollary establishes a characterization of the $\varphi$-directional derivative of the norm in $C_p$-classes ($1 < p < \infty$).

**Corollary 3.1** Let $X, Y \in C_p$. Then, one has

$$D_{\varphi,X}(Y) = pRe \left\{ e^{i\varphi}tr(|X|^{p-1}U^*Y) \right\},$$

for all $\varphi$, where $X = U |X|$ is the polar decomposition of $X$.

**Proof.** Let $X, Y \in C_p$. Put $\tilde{Y} = e^{i\varphi}Y$. Applying Theorem 3.2 with $\varphi, X$ and $\tilde{Y}$ we get

$$D_{\varphi,X}(Y) = \lim_{t \to 0^+} \frac{\|X + te^{i\varphi}Y\|_{C_p} - \|X\|_{C_p}}{t} = \lim_{t \to 0^+} \frac{\|X + t \tilde{Y}\|_{C_p} - \|X\|_{C_p}}{t} = D_X(\tilde{Y})$$

$$= pRe \left\{ tr(|X|^{p-1}U^* \tilde{Y}) \right\} = pRe \left\{ e^{i\varphi}tr(|X|^{p-1}U^*Y) \right\}.$$

This completes the proof. $\square$

Now we are going to characterize the global minimum of $F_\psi$ on $C_p$ ($1 < p < \infty$), when $\phi$ is a linear map satisfying the following useful condition:

$$(3.2) \quad tr(X \phi(Y)) = tr(\phi^*(X)Y), \forall X, Y \in C_p,$$

where $\phi^*$ is an appropriate conjugate of the linear map $\phi$. We state some examples of $\phi$ and $\phi^*$ which satisfy the above condition (3.2).

1. The elementary operator $E : \mathbf{I} \to \mathbf{I}$ defined by

$$E(X) = \sum_{i=1}^{n} A_i X B_i,$$

where $A_i, B_i \in B(H)$ ($1 \leq i \leq n$) and $\mathbf{I}$ is a separable ideal of compact operators in $B(H)$ associated with some unitarily invariant norm. In [8, Proposition 8] the author showed that the conjugate operator $E^* : \mathbf{I}^* \to \mathbf{I}^*$ of $E$ has the form

$$E^*(X) = \sum_{i=1}^{n} B_i X A_i,$$

and that the operators $E$ and $E^*$ satisfy the condition (3.2).

2. Using the previous example we can check that the conjugate operator $\tilde{E} : \mathbf{I}^* \to \mathbf{I}^*$ of the elementary operator $\tilde{E} : \mathbf{I} \to \mathbf{I}$ defined by

$$\tilde{E}(X) = \sum_{i=1}^{n} A_i X B_i - X,$$
Now, we are in position to prove the following theorem.

**Theorem 3.3** Let \( V \in C_p \), and let \( \psi(V) \) have the polar decomposition \( \psi(V) = U|\psi(V)| \). Then \( F_\psi \) has a global minimum on \( C_p \) at \( V \) if and only if \( U^*|\psi(V)| \in \ker \phi^* \).

**Proof.** Assume that \( F_\psi \) has a global minimum on \( C_p \) at \( V \). Then

\[
\inf_{\phi} D_{\phi,\psi(V)}(\phi(Y)) \geq 0,
\]

for all \( Y \in C_p \). That is,

\[
\inf_{\phi} pRe \left\{ e^{i\varphi} tr(|\psi(V)|^{p-1}U^*\phi(Y)) \right\} \geq 0, \forall Y \in C_p.
\]

This implies that

\[
tr(|\psi(V)|^{p-1}U^*\phi(Y)) = 0, \forall Y \in C_p.
\]

Let \( f \otimes g \), be the rank one operator defined by \( x \mapsto \langle x, f \rangle g \) where \( f, g \) are arbitrary vectors in the Hilbert space \( H \). Take \( Y = f \otimes g \), since the map \( \phi \) satisfies (3.2) one has

\[
tr(|\psi(V)|^{p-1}U^*\phi(Y)) = tr(\phi^* (U^*|\psi(V)|^{p-1})Y).
\]

Then (3.4) is equivalent to \( tr(\phi^* (U^*|\psi(V)|^{p-1})Y) = 0 \), for all \( Y \in C_p \), or equivalently

\[
\langle \phi^* (U^*|\psi(V)|^{p-1})g, f \rangle = 0, \forall f, g \in H.
\]

Thus \( \phi^* (U^*|\psi(V)|^{p-1}) = 0 \), i.e., \( U^*|\psi(V)|^{p-1} \in \ker \phi^* \).

Conversely, let \( \varphi \) be arbitrary. If \( U^*|\psi(V)|^{p-1} \in \ker \phi^* \), then \( e^{i\varphi} U^*|\psi(V)|^{p-1} \in \ker \phi^* \). It is easily seen (using the same arguments above) that

\[
Re \left\{ e^{i\varphi} tr(U^*|\psi(V)|^{p-1}\phi(Y)) \right\} \geq 0, \forall Y \in C_p.
\]

Now as \( \varphi \) is taken arbitrary, we get (3.3).

We state our first corollary of Theorem 3.3. Let \( \phi = \delta_{A,B} \), where \( \delta_{A,B} : B(H) \to B(H) \) is the generalized derivation defined by \( \delta_{A,B}(X) = AX - XB \).

**Corollary 3.2** Let \( V \in C_p \), and let \( \psi(V) \) have the polar decomposition \( \psi(V) = U|\psi(V)| \). Then \( F_\psi \) has a global minimum on \( C_p \) at \( V \), if and only if \( U^*|\psi(V)|^{p-1} \in \ker \delta_{U,A} \).

**Proof.** It is a direct consequence of Theorem 3.4.

This result may be reformulated in the following form where the global minimum \( V \) does not appear. It characterizes the operators \( S \) in \( C_p \) which are orthogonal to the range of the derivation \( \delta_{A,B} \).
Theorem 3.4 Let $S \in C_p$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U |\psi(S)|$. Then
\[ \|\psi(X)\|_{C_p} \geq \|\psi(S)\|_{C_p}, \]
for all $X \in C_p$ if and only if $U^*|\psi(S)|^{p-1} \in \ker \delta_{B,A}$.

As a corollary of this theorem we have

**Corollary 3.3** Let $S \in C_p \cap \ker \delta_{A,B}$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U |\psi(S)|$. Then the two following assertions are equivalent:
1. \[ \|S + (AX - XB)\|_{C_p} \geq \|S\|_{C_p}, \text{ for all } X \in C_p. \]
2. $U^*S|^{p-1} \in \ker \delta_{B,A}$.

**Remark 3.1** We point out that, thanks to our general results given previously with more general linear maps $\phi$, Theorem 3.4 and its Corollary 3.3 are true for more general classes of operators than $\delta_{A,B}$ like the elementary operators $E(X)$ and $\sim E(X)$.

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**References**


[3] B.P. Duggal, *Range-kernel orthogonality of the elementary operators $X \to \sum_{i=1}^{n} A_i X B_i - X$*, linear Algebra Appl. 337 (2001), 79-86.


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