

# Extended contraction principle

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## Abstract

Banach's fixed point theorem in fuzzy metric space is studied extensively by many authors. In this article, we are concerned with the implications of one modification of this theorem in the fuzzy metric space  $(X, M, *)$ , with continuous  $t$ -norm  $*$ . In order to do this, first, the concept of chainability in the fuzzy metric space is defined. Then a fixed point theorem for a contractive map,  $[f$  is contractive if  $\frac{1}{M(f(p), f(q), t)} - 1 \leq k(\frac{1}{M(p, q, t)} - 1)$  for a constant  $k]$ , in the chainable fuzzy metric space is proved.

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## 1 Introduction

The concept of fuzzy sets was initially investigated by Zadeh [9] as a new way to represent vagueness in everyday life. There are many examples where the nature of uncertainty in the behavior of a given system possesses fuzzy rather than stochastic nature. Nonstationary fuzzy systems described by fuzzy processes resemble their natural extension into the time domain. They have been carefully studied from different viewpoints .

After Zadeh's pioneering paper [9], the fuzzy sets were developed extensively by many authors and were used in various fields. To use this concept in topology and analysis, several researchers have defined fuzzy metric space in various ways. Kramosil and Michalek [7] defined the fuzzy metric space and then this space was modified by George and Veeramani [2]. This helps us to describe the concept of fuzzy contraction mapping. These maps play a very essential role in introducing the fixed point theory in fuzzy metric spaces. Here, we will prove a fixed point theorem for fuzzy metric space which is chainable. In order to do this, we recall some concepts and findings from [1], [2], [3], [4], [5] and [6] which will be required in the sequel.

**Definition 1.1.** (Jenei and Montagna [5]). A triangular norm ( $t$ -norm) is a binary operation  $T$  on  $[0,1]$  satisfying

- (T1) Commutativity  $T(x, y) = T(y, x)$ ,
- (T2) Associativity  $T(x, T(y, z)) = T(T(x, y), z)$ ,
- (T3) Monotonicity  $T(x, y) \leq T(x, z)$  whenever  $y \leq z$ ,
- (T4) Boundary condition  $T(x, 1) = x$ ,
- (T4') Boundary condition  $T(0, x) = 0$ ,
- (T4'') Conjunctive nature  $T(x, y) \leq \min(x, y)$ .

Note that (T3) and (T4) imply (T4') and (T1), (T3) and (T4) imply (T4'').

**Definition 1.2.** (Schweizer and Sklar [8]). A binary operation  $*$  :  $[0, 1 \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -norm if  $([0, 1], *)$  is a topological monoid with unit 1 such that  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ , and  $a, b, c, d \in [0, 1]$ .

**Definition 1.3.** (George and Veeramani [2]). The 3-tuple  $(X, M, *)$  is said to be a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm, and  $M$  is a fuzzy set on  $X^2 \times ]0, \infty[$  satisfying the following conditions:  
For all  $x, y, z \in X$  and  $t, s > 0$ ,

- (I)  $M(x, y, t) > 0$ ,
- (II)  $M(x, y, t) = 1$  iff  $x = y$ ,
- (III)  $M(x, y, t) = M(y, x, t)$ ,
- (IV)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
- (V)  $M(x, y, \cdot) : ]0, \infty[ \rightarrow [0, 1]$  is continuous.

If  $(X, M, *)$  is a fuzzy metric space, we will say that  $(M, *)$  is a fuzzy metric space on  $X$ .

**Lemma 1.4.** (Grebiec [3]).  $M(x, y, \cdot)$  is nondecreasing for all  $x, y \in X$ .

In order to introduce a Hausdorff topology on the fuzzy metric space, in [2] the authors gave the following definitions:

**Definition 1.5.** (George and Veeramani [2]). Let  $(X, M, *)$  be a fuzzy metric space. The open ball  $B(x, r, t)$  for  $t > 0$  with center  $x \in X$  and radius  $r, 0 < r < 1$ , is defined as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

The family  $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$  is a neighborhood system for a Hausdorff topology on  $X$ .

**Definition 1.6.** (George and Veeramani [2]). In a metric space  $(X, d)$  the 3-tuple  $(X, M_d, *)$  where  $M_d(x, y, t) = t/t + d(x, y)$  and  $a * b = ab$  is a fuzzy metric space. This  $M_d$  is called the standard fuzzy metric induced by  $d$ .

Note that the topologies induced by the standard fuzzy metric and the corresponding metric are the same.

**Theorem 1.7.** (George and Veeramani [2]). A sequence  $(x_n)$  in a fuzzy metric space  $(X, M, *)$  converges to  $x$  if and only if  $M(x_n, x, t) \rightarrow 1$  as  $n \rightarrow \infty$ .

**Definition 1.8.** (George and Veeramani [2]). A sequence  $(x_n)$  in a fuzzy metric space  $(X, M, *)$  is a Cauchy sequence iff for each  $\epsilon \in (0, 1)$ , and each  $t > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1 - \epsilon$  for all  $n, m \geq n_0$ .

A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

With respect to the following lemma, it is easy to verify that a convergent sequence is Cauchy.

**Lemma 1.9.** (Gregori and Sapena [4]). *In a fuzzy metric space  $(X, M, *)$ , for each  $r \in (0, 1)$ , there exists a  $s \in (0, 1)$  such that  $s * s \geq r$ .*

**Definition 1.10.** (Gregori and Sapena [4]). Let  $(X, M, *)$  be a fuzzy metric space. We will say the mapping  $f : X \rightarrow X$  is fuzzy contractive if there exists a  $k \in (0, 1)$  such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right),$$

for each  $x, y \in X$  and  $t > 0$ , ( $k$  is called the contractive constant of  $f$ ).

When  $k = 1$ , the above definition is as follows:

**Definition 1.11.** (Gregori and Sapena [4]). Let  $(X, M, *)$  be a fuzzy metric space. We will say the mapping  $f : X \rightarrow X$  is fuzzy contractive if

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq \left( \frac{1}{M(x, y, t)} - 1 \right),$$

for each  $x, y \in X$  and  $t > 0$ .

**Proposition 1.12.** (Gregori and Sapena [4]). *Let  $(X, M, *)$  be a fuzzy metric space. If  $f : X \rightarrow X$  is fuzzy contractive then  $f$  is  $t$ -uniformly continuous.*

## 2 Extended contraction principal

In this section, first, we will define a fuzzy metric space which is chainable and then we will prove a fixed point theorem.

**Definition 2.1.** Let  $(X, M, *)$  be a fuzzy metric space, we will say the mapping  $f : X \rightarrow X$  is locally fuzzy contractive if for every  $x \in X$  there exists a  $0 < \epsilon < 1$ , and a  $k \in (0, 1)$ , which may depend on  $x$ , such that if

$$p, q \in B(x, y, t) = \{y | M(x, y, t) > 1 - \epsilon\},$$

then

$$\frac{1}{M(f(p), f(q), t)} - 1 \leq k \left( \frac{1}{M(p, q, t)} - 1 \right),$$

for  $t > 0$ , ( $k$  is called the fuzzy contractive constant of  $f$ ).

**Definition 2.2.** Let  $(X, M, *)$  be a fuzzy metric space, a mapping  $f$  of  $X$  into itself is said to be  $(\epsilon, k)$ -uniformly local fuzzy contractive, if it is locally fuzzy contractive and if both  $\epsilon$  and  $k$  do not depend on  $x$ .

**Remark 2.3.** A globally fuzzy contractive mapping can be regarded as a  $(1, k)$ -uniformly local fuzzy contractive mapping.

**Definition 2.4.** Let  $(X, M, *)$  be a fuzzy metric space and  $x, y$  two points of  $X$ , we will say that  $x$  and  $y$  are  $\eta$ -chain, if there is a finite set of points  $x = x_0, x_1, \dots, x_n = y$  and a finite set  $t_0, t_1, \dots, t_{n-1}$  ( $n$  may depend on both  $x$  and  $y$ ) such that  $M(x_{i-1}, x_i, t_{i-1}) > 1 - \eta$ , ( $i = 1, 2, \dots, n$ ,  $0 < \eta < 1$ ).

**Definition 2.5.** The fuzzy metric space  $(X, M, *)$  is said to be  $\eta$ -chainable, if for every  $x, y \in X$  there exists an  $\eta$ -chain.

**Theorem 2.6.** Let  $(X, M, *)$  be a fuzzy metric space, with continuous  $t$ -norm  $*$  defined as  $a * b = \min\{a, b\}$ ,  $a, b \in [0, 1]$ , which is complete and  $\epsilon$ -chainable for all  $\epsilon > 0$ . Also, assume  $f$  is a mapping of  $X$  into itself which is  $(\epsilon, 1)$ -uniformly local fuzzy contractive then there exists a unique point  $\xi \in X$  such that  $f(\xi) = \xi$ .

*Proof:* Let  $x$  be an arbitrary point of  $X$ . Since  $X$  is  $\epsilon$ -chainable, then we can consider the  $\epsilon$ -chain:

$$x = x_0, x_1, \dots, x_n = f(x),$$

where

$$(2.1) \quad M(x_{i-1}, x_i, t_{i-1}) > 1 - \epsilon, \text{ for } i = 1, 2, \dots, n.$$

The property (IV) of Definition 1.3 implies that

$$(2.2) \quad M(x, f(x), t) \geq M(x_0, x_1, t_0) * M(x_1, x_2, t_1) * \dots * M(x_{n-1}, x_n, t_{n-1}),$$

where  $t = t_0 + t_1 + \dots + t_{n-1}$ . Thus the property (T3), and (2.1) give:

$$M(x_0, x_1, t_0) * M(x_1, x_2, t_1) * \dots * M(x_{n-1}, x_n, t_{n-1}) > (1 - \epsilon) * (1 - \epsilon) * \dots * (1 - \epsilon),$$

Therefore (2.2) and definition of  $*$  imply that:

$$(2.3) \quad M(x, f(x), t) > (1 - \epsilon).$$

On the other hand, if  $f$  is  $(\epsilon, 1)$ -uniformly local fuzzy contractive, then for pairs of consecutive points of the  $\epsilon$ -chain, we have:

$$M(f(x_{i-1}), f(x_i), t_{i-1}) > M(x_{i-1}, x_i, t_{i-1}),$$

and with respect to (2.1):

$$M(f(x_{i-1}), f(x_i), t_{i-1}) > (1 - \epsilon),$$

for  $i = 1, 2, \dots, n$ . If we denote  $f(f^m(x)) = f^{m+1}(x)$  ( $m = 1, 2, \dots$ ), then we have by induction and (2.1):

$$(2.4) \quad \begin{aligned} M(f^m(x_{i-1}), f^m(x_i), t_{i-1}) &> M(f^{m-1}(x_{i-1}), f^{m-1}(x_i), t_{i-1}) \\ &> \dots \\ &> 1 - \epsilon. \end{aligned}$$

From the last inequality, we obtain:

$$M(f^m(x), f^{m+1}(x), t) \geq M(f^m(x_0), f^m(x_1), t_0) * M(f^m(x_1), f^m(x_2), t_1) * \dots * M(f^m(x_{n-1}), f^m(x_n), t_{n-1}),$$

where  $t = t_0 + t_1 + \dots + t_{n-1}$ . Now (2.2) and (2.4) will imply:

$$(2.5) \quad M(f^m(x), f^{m+1}(x), t) > (1 - \epsilon) * (1 - \epsilon) * \dots * (1 - \epsilon) = (1 - \epsilon).$$

We now consider  $j$  and  $k$  ( $j < k$ ) to be positive integers, (2.5) gives:

$$\begin{aligned} M(f^j(x), f^k(x), s) &\geq M(f^j(x), f^{j+1}(x), s_j) * M(f^{j+1}(x), f^{j+2}(x), s_{j+1}) \\ &\quad * \dots * M(f^{k-1}(x), f^k(x), s_{k-1}) \\ &> (1 - \epsilon) * (1 - \epsilon) * \dots * (1 - \epsilon) = (1 - \epsilon), \end{aligned}$$

where  $s = s_j + s_{j+1} + \dots + s_{k-1}$ . This shows that  $\{f^i(x)\}$  is a cauchy sequence and thus the completeness of  $X$  guarantees the existence of  $\lim_{i \rightarrow \infty} f^i(x)$ .

From the continuity of  $f$  (clearly implied by Proposition 1.12) it then follows that:

$$f(\lim_{i \rightarrow \infty} f^i(x)) = \lim_{i \rightarrow \infty} f(f^i(x)) = \lim_{i \rightarrow \infty} f^{i+1}(x) = \lim_{i \rightarrow \infty} f^i(x).$$

In order to complete the proof, we have to show that  $\xi = \lim_{i \rightarrow \infty} f^i(x)$  is the only point satisfying  $f(\xi) = \xi$ . Suppose, there exists  $\xi' \neq \xi$  (hence  $M(\xi, \xi', t') < 1$ ) with the property  $\xi' = f(\xi')$  and let  $\xi = y_0, y_1, \dots, y_k = \xi'$  and  $t'_0, t'_1, \dots, t'_{n-1}$  be an  $\epsilon$ -chain, then we have:

$$\begin{aligned} M(f(\xi), f(\xi'), t') &\geq M(f(y_0), f(y_1), t'_0) * M(f(y_1), f(y_2), t'_1) \\ &\quad * \dots * M(f(y_{k-1}), f(y_k), t'_{n-1}) \\ &\geq (1 - \epsilon) * (1 - \epsilon) * \dots * (1 - \epsilon) = (1 - \epsilon) \end{aligned}$$

which is impossible. Hence  $\xi = \xi'$  and the proof is completed.  $\square$

**Remarks 2.7.** Let  $(X, M, *)$  be a fuzzy metric space, with the continuous  $t$ -norm  $*$  defined as  $a * b = \min\{a, b\}$ ,  $a, b \in [0, 1]$ , which is complete and  $\epsilon$ -chainable, and  $f$  a mapping of  $X$  into itself which is  $(\epsilon, 1)$ -uniformly local fuzzy contractive, then there exists a unique point  $\xi \in X$  such that  $f(\xi) = \xi$ .

*Proof:* Note that if  $f$  is  $(\epsilon, k)$ -uniformly local fuzzy contractive then it is  $(\epsilon, 1)$ -uniformly local fuzzy contractive, and Theorem 2.6. completes the proof.  $\square$

## References

- [1] M. A. Erceg. *Metric spaces in fuzzy set theory.* J. Math. Anal. Appl., 69 (1979) 205-230.
- [2] A. George and P. Veeramani. *On some results in fuzzy metric spaces.* Fuzzy Sets and Systems, 64 (1994) 395-399.
- [3] M. Grebiec. *Fixed point in fuzzy metric spaces.* Fuzzy Sets and Systems, 27 (1988) 385-389.

- [4] V. Gregori and A. Sapena. *On fixed point theorems in fuzzy metric spaces*. Fuzzy Sets and Systems, 125 (2002) 245-252.
- [5] S. Jenei and F. Montagna. *A general method for constructing left-continuous  $t$ -norms*. Fuzzy Sets and Systems, 136 (2003) 263-282.
- [6] O. Kaleva and S. Seikkala. *On fuzzy metric spaces*. Fuzzy Sets and Systems, 12 (1984) 215-229.
- [7] O. Kramosil and J. Michalek. *Fuzzy metric and statistical metric space*. Kybernetika, 11 (1975) 326-334.
- [8] B. Schweizer and A. Sklar. *Statistical metric spaces*. Pacific J. Math., 10 (1960) 314-334.
- [9] L. A. Zadeh. *Fuzzy sets*. Inform. Control, 89 (1965) 338-353.

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