

Sum of powers from hyper-pyramids

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Abstract

We transform the problem of finding the sum of powers of integers into a geometrical problem. For this, units are represented by unit cubes in n -dimensional Euclidean spaces and the sums are found by calculating volumes of n -dimensional pyramids.

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Key words: sum of powers, unit cubes, hyper-pyramids.

We derive formulas for the sum of powers of integers from the volumes of "pyramids" in k -dimensional Euclidean spaces. We shall use the word pyramid in a k -D space with its usual definition with symmetry about its height and a volume $V = (1/k)n^{k-1}n$, where n is the length of the edge of the base and also of the height of the pyramid.

The notation "pyramid" will be used for a structure like a pyramid, but constructed from unit k -D cubic blocks.

PYRAMIDS:

$$\left\{ \begin{array}{l} k = 1 : V = (1/1)n^0n = n \\ k = 2 : V = (1/2)n^1n = (1/2)n^2 \\ k = 3 : V = (1/3)n^2n = (1/3)n^3 \\ k = 4 : V = (1/4)n^3n = (1/4)n^4 \\ k = 5 : V = (1/5)n^4n = (1/5)n^5. \end{array} \right.$$

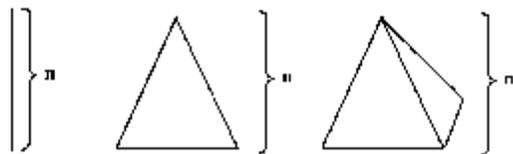


Figure 1.

"PYRAMIDS"

$$k = 1: 1 + 1 + 1 + \dots = n.$$

$$k = 2: 1 + 2 + 3 + \dots + n = \sum_1^n i.$$

$$k = 3: 1 + 2^2 + 3^2 + \dots + n^2 = \sum_1^n i^2.$$

$$k = 4: 1 + 2^3 + 3^3 + \dots + n^3 = \sum_1^n i^3.$$

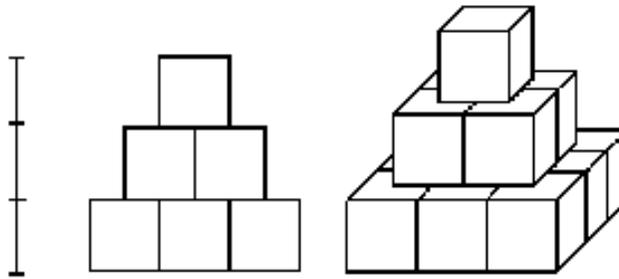


Fig. 2. Fig. 3. Fig. 4.

$$k = 5: 1 + 2^4 + 3^4 + \dots = \sum_1^n i^4.$$

Obviously the volume of the k -D "pyramid" equals the sum of the series. We shall find the volumes to find formulas for the sums. For this we shall look at the lower dimensional cases very carefully. Because, starting with 4-D we lose our visualisation and what we learn from cases of 1-D, 2-D and 3-D will be valuable to find our way in higher dimensional spaces.

VOLUMES OF "PYRAMIDS":

- $k = 1$: $1 + 1 + 1 + \dots = n$ is obvious.

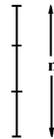


Figure 5.

- $k = 2$.

In 2-D the base of the "pyramid" is a 1-D edge, which has two 0-D corners. We draw from the top of the "pyramid" two lines to these corners. Those lines chop off triangles from the squares of the sides. The 2-D "volume" of each chopped off triangle is: $\frac{1}{2} \left(\frac{1}{2}\right) (1)$.

In higher dimensional spaces this volume will be generalized to $\frac{1}{2} \left(\frac{1}{2}\right) (1)(1)$ for 3-D;

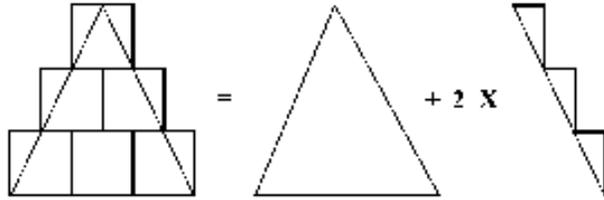


Figure 6.

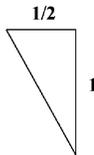


Figure 7.

$\frac{1}{2} \left(\frac{1}{2}\right) (1)(1)(1)$ for 4-D, etc.

Here on each side there are $1 + 1 + 1 + \dots = n$ of them. That is they form the sequence of the one-less-dimension. Thus the volume of the "pyramid" equals the volume of the pyramid plus the chopped off volumes.

$$\sum_1^n i = 1 + 2 + 3 + \dots + n = \frac{1}{2}n^2 + 2n\frac{1}{2} \left(\frac{1}{2}\right) (1) = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n + 1)$$

- $k = 3$.

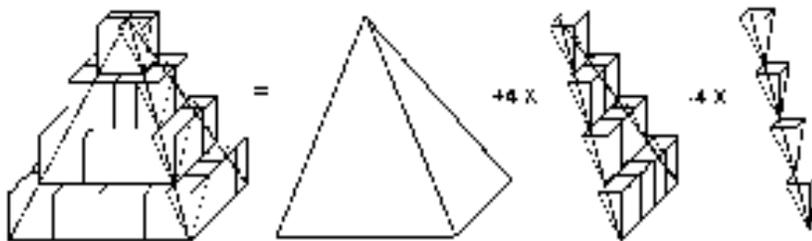


Figure 8.

Here the base is a 2-D plane with 4 1-D edges and 4 0-D corners. We draw from the top, 4 2-D planes which go through the 4 1-D edges. Those planes chop off from the 3-D cubes on the faces, pieces which look like

These 3-D pieces have volumes $\frac{1}{2} \left(\frac{1}{2}\right) (1)(1)$, as we anticipated in $k = 2$ case. On each side the number of these pieces form the sequence $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1)$. This is the sequence in the one-less-dimensional case. There are four faces. Hence we

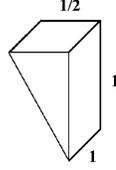


Figure 9.

must add to the volume $\frac{1}{3}n^3$ of the pyramid

$$\frac{1}{2} \left(\frac{1}{2} \right) (1)(1) 4 \frac{1}{2} n(n+1) = \frac{1}{2} n(n+1).$$

However, where the 4 edges meet at the 4 corners we are double counting the contributions from two adjacent chopped off pieces.

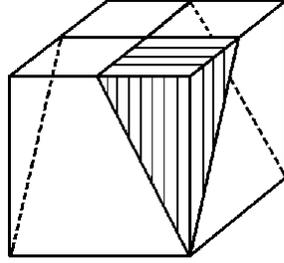


Figure 10.

These double-counted small pyramids have a 3-D volume of

$$\frac{1}{3} \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (1) = \frac{1}{12}.$$

Their counterparts in higher dimensional spaces will be

$$\frac{1}{3} \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (1)(1)$$

in 4-D;

$$\frac{1}{3} \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (1)(1)(1),$$

in 5-D; etc. These double-counted pieces form from each corner of the base to the top of the pyramid a sequence $1 + 1 + 1 + \dots + 1 = n$. This is the sequence in the two-less-dimensional space. There are 4 such sequences, since there are 4 0-D corners. We must now subtract the volume of these double-counted small pyramids, which is $4n(\frac{1}{12}) = \frac{n}{3}$. The total volume of the "pyramid" becomes:

$$\sum_1^n i^2 = 1 + 4 + 9 + \dots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n(n+1) - \frac{n}{3} = \frac{1}{6}n(n+1)(2n+1).$$

- $k = 4$.

In 4-D space we lose our geometric visualization. However what we learned from 1-D, 2-D, and 3-D, spaces will help us. We will start with the generation of a k -D cube from a $(k - 1)$ -D cube starting with the 1-D case.

In 1-D, a cube has one edge and 2 0-D corners. When we add a new dimension, to generate a cube in 2-D space we shift the 1-D cube orthogonally to itself into the new dimension. The old 0-D corners come to their new locations. Thus their number is doubled. The old 1-D edges come to their new locations and their number also is doubled. But the one less dimensional corners trace two new edges, so that the number of the edges of the 2-D cube is four.

This process is general for all features of a k -D cube, except for the 0-D corners, which simply double. That is, the number of the features is doubled and to this we must add the number of the one less dimensional feature. (See Fig. 11)

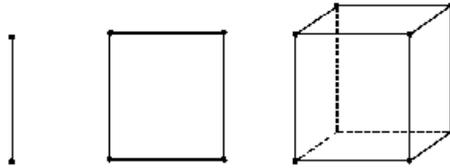


Figure 11.

We give below a table for the features of k -D cubes up to 6-D. The rule stated above is easily checked from this table.

	0-D corner	1-D edge	2-D side	3-D side	4-D	5-D	6-D
1-D	2	1	-	-	-	-	-
2-D	4	4	1	-	-	-	-
3-D	8	12	6	1	-	-	-
4-D	16	32	24	8	1	-	-
5-D	32	80	80	40	10	1	-
6-D	64	192	240	160	60	12	1

What makes it possible for us to make the transition from 3-D to 4-D space are the following observations:

1. The base of the k -D pyramid is a $(k - 1)$ -D hyperplane.

The base has a certain number of $(k - 2)$ -D features. We put through those $(k - 2)$ -D features hyperplanes of $(k - 1)$ -D which chop off pieces of volume $\frac{1}{2} (\frac{1}{2})(1)(1)$ of k -D. These pieces form a sequence of the $(k - 1)$ space. The number of these sequences equals the number of the $(k - 2)$ -D features of the k -D cube.

2. Everytime we raise to a higher dimensional space a new feature emerges. This feature introduces a double counted pyramid starting with $\frac{1}{3} (\frac{1}{2}) (\frac{1}{2})(1)$, which form sequences of the $(k - 2)$ -D space. Their number equals the number of the $(k - 2)$ -D features of the k -D cube. In higher D's these volumes become $\frac{1}{3} (\frac{1}{2}) (\frac{1}{2}) (1)(1) \dots$

3. This pattern continues with the next double counted pyramid being $\frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)(1)$. Everytime such a sequence appears it forms the sequence $\sum i^0$. In the next higher D space its sequence becomes $\sum i$ and so on.

A 4-D pyramid has a volume of $V = \frac{1}{4}n^4$.

We will calculate the volume of a 4-D "pyramid" built in a 4-D space with unit 4-D cubes.

The height of the "pyramid" is n . Its base is a 3-D hyperplane with a 3-D "area" of n^3 .

This 3-D base has:

0-D corners; 12 1-D edges; 2-D sides.

We draw from the top of the 4-D "pyramid" 6 3-D hyperplanes, which go through the 6 2-D sides of the base. They chop off from the 4-D cubes on each of the 6 faces pieces with volumes

$$\frac{1}{2} \left(\frac{1}{2}\right) (1)(1)(1) = \frac{1}{4},$$

as we have anticipated in the $k = 2$ case. Their number on each face forms the one-less-dimensional sequence

$$1 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Since there are 6 faces, the total chopped off 4-D volume becomes

$$(0.1) \quad \frac{1}{4}6(1/6)n(n+1)(2n+1) = \frac{1}{4}(2n^3 + 3n^2 + n).$$

We must add this volume to the volume $\left(\frac{1}{4}n^4\right)$ of the pyramid. However those 6 2-D sides intersect at 12 1-D edges. Here there is double counting of the intersections of the 4-D cubes. The common double counted volume is

$$\frac{1}{3} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) (1)(1) = \frac{1}{12},$$

as we anticipated in the $k = 3$ case.

On each of the 12 2-D sides, they form base to the top, the two-less-dimensional sequence

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1).$$

There are 12 such sequences, with each unit having a volume of $1/12$. Hence we must subtract from the sum

$$\frac{1}{4}n^4 + \frac{1}{4}(2n^3 + 3n^2 + n)$$

the double counted volume

$$12 \left(\frac{1}{12}\right) \frac{1}{2}n(n+1) = \frac{1}{2}(n^2 + n).$$

But now these 12 2-D sides intersect at 8 0-D corners, where the common volume of a small 4-D pyramid is

$$\frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) (1) = \frac{1}{32}.$$

The analogue of this in 5-D space will be

$$\frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) (1)(1) = \frac{1}{32}.$$

These chopped off volumes form the three-less-dimensional sequence on each of 8 edges $1 + 1 + 1 + \dots + 1 = n$. There are 8 such sequences. Thus we have over-compensated the total volume and we must add now $8n\frac{1}{32} = \frac{1}{4}n$.

Finally the total volume of the 4-D "pyramid" becomes

$$\begin{aligned} \sum_1^n i^3 &= 1 + 2^3 + 3^3 + \dots + n^3 \\ &= \frac{1}{4}n^4 + \frac{1}{4}(2n^3 + 3n^2 + n) - \frac{1}{2}(n^2 + n) + \frac{1}{4}n \\ &= \frac{1}{4}n^2(n+1)^2. \end{aligned}$$

• $k = 5$.

A 5-D pyramid has a volume of $V = (1/5)n^5$. We will calculate the volume of a 5-D "pyramid" built in a 5-D space with unit 5-D cubes. The height of the "pyramid" is n . Its base is a 4-D hyperplane with a 4-D "area" of n^4 . This 4-D base has:

0-D corners; 1-D edges; 2-D sides; 8 3-D sides

We draw from the top of the 5-D "pyramid" 8 4-D planes which go through the 8 3-D sides of the base. These "planes" chop off from the 5-D cubes on each of the 8 faces, pieces, each with a volume $\frac{1}{2} \left(\frac{1}{2}\right) (1)(1)(1) = \frac{1}{4}$ as we anticipated in $k = 2$ case. Their number on each face forms the one-less-dimensional sequence

$$1 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

Since there are 8 faces, the total chopped off 5-D volume becomes

$$\frac{1}{4}8\frac{1}{4}n^2(n+1)^2 = \frac{1}{2}n^2(n+1)^2.$$

We must add this volume to the volume $(1/5)n^5$ of the pyramid.

However those 8 4-D hyperplanes through the 8 3-D sides of the base intersect at 24 2-D sides. Thus there is double counting at the intersections of the 5-D cubes. The common intersection volumes are

$$\frac{1}{3} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) (1)(1) = \frac{1}{12}$$

as we anticipated in $k = 3$ case. On each of the 24 2-D sides, they form from base to the top the two-less-dimensional sequence

$$1 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n).$$

There are 24 such sequences, with each unit having the volume of $1/12$. Hence we must subtract from the sum

$$\frac{1}{5}n^5 + \frac{1}{2}n^2(n+1)^2$$

the double counted volume

$$24 \left(\frac{1}{12} \right) \left(\frac{1}{6} \right) (2n^3 + 3n^2 + n) = \frac{1}{3} (2n^3 + 3n^2 + n).$$

But now the 24 2-D sides intersect at 32 1-D edges. The common volume at these intersections is

$$\frac{1}{4} \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (1)(1) = \frac{1}{32},$$

as we anticipated in $k = 4$ case.

These form from the base to the top, the three-less-dimension sequence

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1).$$

Since there are 32 such sequences, the total over subtracted volume becomes

$$32(1/32)\frac{1}{2}n(n + 1) = \frac{1}{2}n(n + 1).$$

This, we must add to our previous result. Finally, the 24 2-D sides intersect at 16 corners. At those intersections the volumes of the common small pyramids are

$$\frac{1}{5} \left(\frac{1}{2} \right)^4 (1).$$

They form from the base to the top, the sequence $1 + 1 + 1 + \dots + 1 = n$. Since there are 16 of those sequences, the over-compensated volume becomes

$$16\frac{1}{5} \left(\frac{1}{2} \right)^4 (1) = \frac{1}{5}n.$$

This we must subtract from our previous result. Combining our results we find the total volume of the 5-D "pyramid".

$$\sum_1^n i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^2(n + 1)^2 - \frac{1}{3}(2n^3 + 3n^2 + n) + \frac{1}{2}n(n + 1) - \frac{1}{5}n;$$

$$\sum_1^n i^4 = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n).$$

We applied the process up to 5-D. Obviously, after having learned how to calculate the double counted volumes, when a k -dimensional "pyramid" is cut with $k-1$ dimensional hyperplanes from the top to the $k-2$ dimensional "edges" of the base, we can find the sum of any series with power $k-1$. Denoting the sum indexed by $i = \overline{1, n}$ briefly by Σ , we have,

0-D

$$\sum i = \frac{1}{2}n^2 + 2 \sum i^0 \frac{1}{4} = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n + 1)n^2$$

$$\sum i^2 = \frac{1}{3}n^3 + \overset{1\text{-D}}{4} \sum i^{\frac{1}{4}} - \overset{0\text{-D}}{4} \sum i^0 \frac{1}{12} = \frac{1}{3}n^3 + \frac{1}{2}n(n+1) - \frac{1}{3}n = \frac{1}{6}n(n+1)(2n+1).$$

$$\begin{aligned} \sum i^3 &= \frac{1}{4}n^4 - \overset{2\text{-D}}{6} \sum i^2 \frac{1}{4} + \overset{1\text{-D}}{12} \sum i^{\frac{1}{12}} + \overset{0\text{-D}}{8} \sum i^0 \frac{1}{32} = \\ &= \frac{1}{4}n^4 + \left(\frac{3}{2}\right) \left(\frac{1}{6}\right) n(n+1)(2n+1) - \frac{1}{2}n(n+1) + \frac{1}{4}nn^4 \\ &\quad \sum i^3 = \frac{1}{4}n^2(n+1)^2 \end{aligned}$$

$$\begin{aligned} \sum i^4 &= \frac{1}{5}n^5 + \overset{3\text{-D}}{8} \sum i^3 \frac{1}{4} - \overset{2\text{-D}}{24} \sum i^2 \frac{1}{12} + \overset{1\text{-D}}{32} \sum i^{\frac{1}{32}} - \overset{0\text{-D}}{16} \sum i^0 \frac{1}{8} = \\ &= \frac{1}{5}n^5 + \frac{1}{2}n^2(n+1)^2 - 2\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - \frac{1}{5}n = \\ &= \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n). \end{aligned}$$

$$\begin{aligned} \sum i^5 &= \frac{1}{6}n^6 + \overset{4\text{-D}}{10} \sum i^4 \frac{1}{4} - \overset{3\text{-D}}{40} \sum i^3 \frac{1}{12} + \overset{2\text{-D}}{80} \sum i^2 \frac{1}{32} - \overset{1\text{-D}}{80} \sum i^{\frac{1}{80}} + \overset{0\text{-D}}{32} \sum i^0 \frac{1}{192} \frac{1}{12} = \\ &= \frac{1}{6}n^6 + \frac{5}{2} \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n) - \frac{10}{3} \frac{1}{4}n^2(n+1)^2 + \\ &\quad + \frac{5}{2} \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) + \frac{1}{6}n = \frac{1}{12}n^2(2n^4 + 6n^3 + 5n^2 - 1). \end{aligned}$$

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