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PRESENTATION

This thesis is devoted to the study the Dirichlet problem for some nonlinear elliptic equation within a non variational framework.

This works is divided into two principal parts. Each part is preceded by an introduction and a detailed summary which starts with a general introduction.

In the first part, we investigate the existence of a solution of the following nonlinear elliptic problem:

$$Au + g(x, u, \nabla u) - \text{div}\phi(u) = \mu,$$

where $Au = \text{div}(a(x, u, \nabla u))$ is the Leray-Lions operator, the function $g(x, u, \nabla u)$ is a nonlinear lower order term with natural growth, while ϕ is a continuous function from $I\!R$ into $I\!R^N$. Two different kind of questions have been considered:

On one hand, we focused on the existence of solutions of the unilateral problem in the context of Orlicz Sobolev spaces, with the source $\mu \in L^1$ or L^1-dual and the nonlinearity term satisfying the classical sign condition $g(x, s, \xi).s \geq 0$.

On the other hand, in the same context of Orlicz Sobolev spaces, we focus on the study of the unilateral problem associated with the above equation with $\phi = 0$ without any sign condition on g.

The second part is devoted to the study of the strongly nonlinear elliptic problem (degenerate or singular) in a broader context, that of weight Sobolev spaces.

Contents

General Introduction

The study of elliptic equations and inequations proved very important given their application in various fields of physics, biology, astronomy,...
The case of nonlinear elliptic equations with second order of Leray-Lions type i.e.

$$Au = \operatorname{div}(a(x, u, \nabla u)) = f, \tag{0.0.1}$$

with different boundary conditions, has been the subject of numerous studied since the early fifties years. There are for examples the work of Visik [110, 111], Browder [63], Morrey [95], Nirenberg [100] and others.

For the variational case, i.e., when the second member f is in the dual $W^{-1,p'}(\Omega)$ $p > 1$, the nonlinear elliptic equation of the form (0.0.1), have been the subject of an initial work given by Leray and Lions [93] in the early sixties, where they proved the existence of a weak solution of variational problem (associated with the classical Dirichlet problem) as follows :

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla v \, dx \leq \langle f, v\rangle, \quad \forall \;\; v \in W_0^{1,p}(\Omega). \end{cases} \tag{0.0.2}$$

When a does not depend on u, there is uniqueness of the solution. In more general situation where a is a Lipshitz function with respect u, Boccardo, Murat and Gallouët [51] showed the uniqueness of solution of (0.0.2) for $p \leq 2$, and there are counter-example for the uniqueness for $p > 2$.
If we consider the elliptic problem where the data are no longer in $W^{-1,p'}(\Omega)$, the above formulation not adapted. For $f \in \mathcal{M}(\Omega) = (\mathcal{C}(\overline{\Omega}))'$, the space of measures, Stampacchia [108] proposed in 1965 a method giving, in the lineare case. i.e.

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega (A\nabla u)\nabla v \, dx \leq \langle f, v\rangle, \\ \forall \;\; v \in H_0^1(\Omega) \;\; \text{where A is a bounded and coercive matrix,} \end{cases} \tag{0.0.3}$$

the existence and uniqueness of solution of this elliptic equation.
This method uses the duality (and a regularity result) and leads to variational formulation of type :

$$\begin{cases} u \in \displaystyle\bigcap_{q < \frac{N}{N-1}} W_0^{1,q}(\Omega), \\ -\langle \operatorname{div}(\mathrm{A}^{\mathrm{t}}\nabla \mathrm{u}), v\rangle = \displaystyle\int_\Omega v \, df, \\ \forall \;\; v \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) \\ \text{such that } \; \operatorname{div}(\mathrm{A}^{\mathrm{t}}\nabla \mathrm{v}) \in \displaystyle\bigcup_{r > N} W^{-1,r'}(\Omega). \end{cases} \tag{0.0.4}$$

the weak classic formulation is would rather

$$\begin{cases} u \in \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega), \\ \int_\Omega A^t \nabla u \nabla v \, dx = \int_\Omega v \, df \quad \forall \;\; v \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) \;\text{ such that }\; \operatorname{div}(A^t \nabla v) \in \bigcup_{r>N} W_0^{1,r}(\Omega). \end{cases} \tag{0.0.5}$$

Since $W_0^{1,r}(\Omega) \subset \mathcal{C}(\overline{\Omega})$, and for all $v \in W_0^{1,r}(\Omega)$, we have $\mathrm{div}(\mathrm{A^t \nabla v}) \in W^{-1,r'}(\Omega)$, the test function of (0.0.5) my be also chosen as a test function of (0.0.4), but the reciprocal is false.
Also (0.0.5) does not ensure the uniqueness of solutions, while (0.0.4) assured.
Always for $f \in \mathcal{M}(\Omega)$, but with nonlinear operator (for example the nonlinear elliptic equation (0.0.1)), the Stampacchia method deos not applicable. A little more after in 1989, Boccardo and Gallouët [48] built by approximation a solution to the equation with homogenous Dirichlet boundary condition. The formulation is similar to this of distributions :

$$\begin{cases} u \in \bigcap_{q<\frac{N(p-1)}{N-1}} W_0^{1,q}(\Omega), \\ \int_\Omega a(x, u, \nabla u) \nabla v \, dx = \int_\Omega v \, df, \;\; \forall \;\; v \in \bigcup_{r>N} W_0^{1,r}(\Omega). \end{cases} \tag{0.0.6}$$

In the case where a does not depend on u, the existence of a solution is firstly given in [48] and faster in [55], with a technical assumption on a in order to have the almost everywhere convergence of ∇u_n to ∇u. This condition was delated in [49].
Concerning the case of homogenous Dirichlet elliptic boundary condition, where data in $L^1(\Omega)$ (which is of course more restrictive than $\mathcal{M}(\Omega)$), three approach were used.
Dall'Aglio [64] has shown that even for a nonlinear problem, the method by approximation leads to a one solution called SOAL (solution obtained by approximations limit).
In [34], Bénilan, Boccardo, Gallouët, Gariepy, Pierre and Vazquez, define the notion of entropy solution (here $p > 2 - \frac{1}{N}$ for simplify),

$$\begin{cases} u \in W_0^{1,1}(\Omega), T_k(u) \in W_0^{1,p}(\Omega) \\ \int_\Omega a(x, \nabla u) \nabla T_k(u - v) \, dx \leq \int_\Omega f T_k(u - v) \, dx, \;\; \forall \;\; v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{0.0.7}$$

For more general homogenous boundary conditions and in particular Neumann and Fourier, the existence and uniqueness of entropy solution was shown by Andreu, Mazon, Segura de Leon and Toledo [22].
Finally Lions and Murat [94, 96] have introduced another concept that of renormalized solution (following the same name of solutions due to Di Peurna and Lions for the

Boltzman equation). It verifies (here also, $p > 2 - \frac{1}{N}$) :

$$\begin{cases} u \in W^{1,1}(\Omega), T_k(u) \in W_0^{1,p}(\Omega) \quad \forall k > 0 \\ \lim\limits_{h\to\infty} \int_{\{h\le|u|\le k+h\}} |\nabla u|^p \, dx = 0 \quad \forall \; k > 0 \\ \int_\Omega S(u)a(x,\nabla u)\nabla v \; dx + \int_\Omega S'(u)va(x,\nabla u)\nabla u \; dx = \\ \qquad\qquad\qquad \int_\Omega fS(u)v \; dx, \quad \forall \;\; v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{0.0.8}$$

for any regular real variable function S, with compact support.
The uniqueness criteria (SOAL, entropy and renormalized solution) usable for $f \in L^1(\Omega)$, because $T_k(u)$ and $S(u)v \in L^\infty(\Omega)$, however the uniqueness of entropy solution was extend by Boccardo, Gallouët and Orsina [53] to case of second member measure does not charge of p-capacity zero, i.e. for the second member μ verified, for a Borelien B :

$$Cap_p(B, \Omega) = 0 \Rightarrow \mu(B) = 0.$$

This equivalent , (see [53]) for a measure $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$.
Where the second member is any measure, we do know in general shown that, the concept of SOAL ensures the uniqueness, and the entropy and renormalized solutions are no longer defined.
For the elliptic equations, new equivalent definitions have been proposed by Dal Maso, Murat, Orsina and Prignet [66], they generalize the previous three concepts, but do not also to uniqueness.
Consider now the equation (0.0.1), but with the introduction of a nonlinear term $g(x, u)$, depending on x and u, i.e. an equation called strongly nonlinear of type :

$$(Au = -\text{div}(a(x, u, \nabla u))) + g(x, u) = f. \tag{0.0.9}$$

The study of this type of problem was firstly treated by F. E. Brower [63] by using the theory of non bounded monotone operators in Sobolev spaces $W^{m,p}(\Omega)$. after a study of this kind was made by P. Hess [85], by using a regular enough truncation in the case $m = 1$ and under weaker conditions. Afterward, J. Webb, H. Brezis and F. E. Browder in [112, 61] have studied the problem (0.0.9) (with a second member in the dual $W^{-m,p'}(\Omega)$) in the case of higher order by using a Hedberg approximation type [83], mentien also the work of Boccardo, Giachetti and Murat [57] which generalize that of Brezis-Browder [61].The result obtained is used to show the existence of solutions for some unilateral problems.
When the nonlinearity g depends on three variables $x, u, \nabla u$, our equation becomes :

$$(Au = -\text{div}(a(x, u, \nabla u))) + g(x, u, \nabla u) = f, \tag{0.0.10}$$

with g satisfies the natural growth condition on $|\nabla u|$ and the classical sign condition $g(x, s, \xi).s \geq 0$, we found in the variational case (i.e. $f \in W^{-1,p'}(\Omega)$), the work of Bensoussan, Boccardo and Murat [44], where the existence of a weak solution of the strongly nonlinear problem associated to (0.0.10) was given by using the strong convergence of the positive (resp. negative) part of the sequence u_ε of the approached

problem.
The same problem was studied by Boccardo, Gallouët and Murat [52] by using the strong convergence of truncations.
The extension of these results when $f \in L^1(\Omega)$ was treated in particular in [50],but under a coercivity condition on the nonlinearity of type:

$$|g(x,s,\xi)| \geq \gamma|\xi|^p \quad \text{for} \quad |s| \geq \eta. \tag{0.0.11}$$

The use of (0.0.11) appears in particularly at level of $u \in W_0^{1,p}(\Omega)$.
Next, Porretta [102] delated the condition (0.0.11) and proved the existence of solution of the strongly nonlinear problem associated to (0.0.10) with $u \in W_0^{1,q}(\Omega)$ for all $q < \frac{N(p-1)}{N-1}$.
More recently, the same author [103] studied the problem (0.0.10) without assuming the sign condition, but by changing the classical growth condition of the nonlinearity g by

$$|g(x,s,\xi)| \leq h(x) + \rho(s)|\xi|^p, \quad \rho \in L^1(\Omega), \rho \geq 0. \tag{0.0.12}$$

The unilateral case is also treated in our work [10] without assuming the sign condition.
Throughout the work cited earlier, the growth of the operator A and the nonlinearity g are of polynomial type, i.e

$$|a(x,s,\xi)| \leq k(x) + |s|^{p-1} + |\xi|^{p-1}, \tag{0.0.13}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^p), \tag{0.0.14}$$

also the conditions of coercivity of A and g have the form,

$$a(x,s,\xi)\xi \geq \alpha|\xi|^p, \tag{0.0.15}$$

$$|g(x,s,\xi)| \geq \gamma|\xi|^p \quad \text{for} \quad |s| \geq \eta. \tag{0.0.16}$$

this required to formulate all the problems mentioned above in a classical functional, that Sobolev spaces $W^{m,p}(\Omega)(m \geq 1)$.
When the coefficients of A (resp. g) are relaxed, i.e. the growth and coercivity are non-polynomial size (replace for example $|t|^p$ by the N-function $M(t)$), the study of elliptic problem (strongly nonlinear and unilateral) associated to different equations (0.0.1), (0.0.9) et (0.0.10), requires a more general functional framework which of Orlicz Sobolev spaces $W^1L_M(\Omega)$.
The problem of type (0.9) with $g = 0$ have been solved by J. P. Gossez [77, 78]; while the solving of problems (0.0.9) with ($g \neq 0$), it has been studied by J. P. Gossez [79, 81] and by A. Benkirane - J. P. Gossez [42] int the case of higher order.
The problem of type (0.0.10) have been solved by A. Benkirane-A. Elmahi [39, 40] where the datum in $L^1(\Omega)$ or dual. These results were obtained under the assumption that the N-function satisfies the Δ_2-condition near infinity, the extension of these results for the general N-functions can be found in the work of D. Meskine - A. Elmahi [72, 73].
Concerning the unilateral case with datum in L^1, we have treated in the first case in [5],

with a constraint on the N-function which is the Δ_2-condition and an obstacle on the positive part in $W_0^1L_M(\Omega)\cap L^\infty(\Omega)$. In a second case, the same problem was restudied in [8, 7] with the source term in L^1 or dual, but illuminating of two fundamental assumptions, the first is the Δ_2-condition and the second is the regularity on the obstacle ψ (i.e. $\psi^+ \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$).
On the other side, A. Benkirane and J. Bennouna [38, 36] have studied the problem of type

$$-\mathrm{div}(a(x,u,\nabla u)) - \mathrm{div}\phi(u) = f, \tag{0.0.17}$$

(where$\phi \in \mathcal{C}^0(I\!R, I\!R^N)$), in the context of Orlicz spaces with $\phi = 0$ for treated the unilateral case associated to (0.0.17) and $\phi \neq 0$ for the case of an equation where an entropy solution have proved.
A generalization of (0.0.17) in the unilateral case in the sense of an obstacle free was the major objective of my work [12, 13].
In this direction, we find the study of existence and in particular the regularity solutions of the poisson equation :

$$\Delta u = f$$

In the work of Azroul, Benkirane and Tienari [32], where the regularity is studied in different cases of the second member f (f measure, distribution of order 1, an element of Orlicz spaces), by using the Newton potential and the Torschinsky interpolation. The results obtained in [32] gives in particular a refinement of the L^p case where ($p < \frac{N}{N-1}$ or $p < \frac{N}{N-2}$). In this sense we found the results of [35, 3, 6].
On the other hand, if the condition of ellipticity (0.0.15) is replaced by

$$a(x,s,\xi)\xi \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi|^p, \tag{0.0.18}$$

where $w = \{w_i(x), i = 1,...,N\}$, is a family of functions defined and positive almost everywhere on Ω, but are not separated from zero (called weight functions), then the differential operator becomes degenerate. In this case, it is brought to change the traditional approach by introducing a modified version of the Sobolev spaces, called weight Sobolev spaces $W^{1,p}(\Omega, w)$. These spaces presented the functional framework of a part of our work. Note that the major difficulties en countered during the study of degenerate problem appear at of topological properties of weight Sobolev spaces, such as compact imbedding and the type of convergence. The approach of degenerate problems was introduced initially by M. K. Murthy and G. Stampachia [97] in the case of a linear operator of second order. An extension to the case of linear degenerate elliptic operator of higher order was treated by several authors, for examples, V. P. Glushko [76], A. Kufner and B. Opic [89]. For degenerate nonlinear operator, we found the work of J. P. Rakotoson [104, 105] in which the authors has studied the degenerate problem of type

$$Au + F(u, \nabla u) = f$$

and variational inequalities associated with the concept of relative rearrangement. We refer also the reader to [23, 24, 25]. Later, F. Gugliemino and F. Nicolosi [82] have shown in the case $p = 2$ the existence of a weak solution of the equation $Au = f$ where

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + c_0 |u|^{p-2} u + a_0(x, u, \nabla u)$$

by using another type of degeneration.

$$\sum_{i=1}^{N} a_i(x, \eta, \xi)\xi \geq \frac{\rho(x)}{k(|\eta|)} \sum_{i=1}^{N} |\xi_i|^p,$$

where $\rho(x)$ is a weight function on Ω and h is a function satisfying certain assumptions. This result was generalized to the case $p > 1$ by P. Drabek and F. Nicolosi in [70] relying on a special truncation and the Lreay-Lions theorem.

In 1994, P. Drabek, A. Kufner and F. Nicolosi in [68] (see also [70]) have studied the nonlinear degenerate elliptic problem if higher order generated by the operator,

$$Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, ..., \nabla^m u),$$

by using the theory of degree. In 1998, P. Drabek, A. Kufner and V. Mustonen in [67] have studied the nonlinear degenerate elliptic problem associated to an operator without lower order term of type

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u),$$

by using the large monotony condition. After Azroul [29] a generalized this result to case of higher order by using the large monotony in the case where A has no lower order terms and the strict monotony in the case where A is a lower order terms.

In $2001 - 2002$ Akdim, Azroul and Benkirane studied in [17, 20] the strongly nonlinear degenerate problem associated to equation (0.0.10) by taking the second member f in the dual or in L^1 and with a nonlinearity $g(x, u, \nabla u)$ satisfying a sign condition and having a natural growth with respect to ∇u. The approach used in the first work is that of sup and sub solutions while in the second is the strong convergence of truncations in weight Sobolev spaces.

Note that the authors in [17, 20] imposed the Hardy inequality, i.e.

(H) there exists a weight function σ in Ω and a parameter $q, 1 < q < \infty$, such that

$$\left(\int_{\Omega} |u|^q \sigma(x)\, dx\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{N} \int_{\Omega} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}, \tag{0.0.19}$$

for all $u \in W_0^{1,p}(\Omega, w)$ and where C is a positive constant independent of u, and a condition of integrability on the weight of Hardy type,,

$$\sigma^{1-q'} \in L^1_{loc}(\Omega). \tag{0.0.20}$$

Moreover in the case of second member L^1, the authors added the following coercivity condition

$$|g(x,s,\xi)| \geq \gamma \sum_{i=1}^{N} w_i |\xi_i|^p \quad \text{for} \quad |s| \geq \eta. \tag{0.0.21}$$

More recently, we found in $2003-2004$ the work of these authors who deal with a degenerated unilateral problem associated to equation (0.0.10) with the second member f in $W^{-1,p'}(\Omega, w)$.
There are still many questions concerning the degenerated problem that must be treated, among these questions i quote the work [4, 9] in which we have studied the degenerated strongly nonlinear unilateral problem mentioned above in the case $(f \in L^1(\Omega))$, but without any coercivity on the disruptive term g (as (0.0.21)).
Note that, in our work, the Hardy parameters $\sigma(x)$ and q are not any but respond to such assumptions, that

$$\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \quad 1 < q < p + p' \tag{0.0.22}$$

The latter condition involved more specially in the demonstration of existence of at least one solution of the approached problem and their estimate. For surmounted the difficulty we has changed the coercivity

$$a(x,s,\xi)\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p$$

by another of type

$$a(x,s,\xi)(\xi - \nabla v_0) \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p - \delta(x),$$

where $v_0 \in K_\psi \cap L^\infty(\Omega), \delta \in L^1(\Omega)$ and introduced an approximation of the nonlinearity g of the form

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}\theta_n(x)$$

with $\theta_n(x) = nT_{1/n}(\sigma^{1/q}(x))$. So we got take Hardy parameters without any restriction and is exactly the content of the work [15].
Note that the case of degenerated elliptic problems of higher order is studied in [21] under general growth condition and where the monotony condition is assumed on part of the operator while on the other part is strictly monotone.
If always we keep the degeneration, but the coefficients of the operator satisfies the growth condition of Orlicz type, the functional framework which comes into question for formulating our problem is a more general framework which is the weight Orlicz Sobolev spaces. It is important to recall a single work can be found in this directions that in 2000 by E. Azroul [31] where in particular a result of compact embedding is shown.

Part I

STUDY OF SOME UNILATERAL PROBLEMS

This part is constituted of the following Chapters:

Chapter I

Preliminaries

Chapter II

Existence of solutions for unilateral problems with L^1 data in Orlicz spaces

Chapter III

Strongly nonlinear elliptic unilateral problems with natural growth in Orlicz spaces

Chapter IV

Existence results for some unilateral problems without sign condition in Orlicz spaces.

Chapter V

Existence of solutions for unilateral problems in L^1 involving lower order terms in divergence form in Orlicz spaces

Introduction and summary of the first part

In the first part we was interested in the existence of solutions for some nonlinear elliptic problems in Orlicz-Sobolev spaces.

I.1 Existence of solutions for unilateral problems with L^1 data in Orlicz spaces.

Consider the following nonlinear Dirichlet problem

$$Au + g(x, u, \nabla u) = f, \tag{0.1}$$

where A is Leray-Lions operator defined on its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ in to $W^{-1}E_{\overline{M}}(\Omega)$, while g is a nonlinearity with the following "natural" growth condition

$$|g(x, s, \xi)| \leq b(|s|)\,(c(x) + M(|\xi|))$$

and which satisfies the classical sign condition $g(x, s, \xi).s \geq 0$. The right hand side f is assumed to belong to $L^1(\Omega)$.
In the work [79], Gossez have establish an existence result for problems of the form (0.1) when $g \equiv g(x, s)$ and $f \in W^{-1}E_{\overline{M}}(\Omega)$.
The solvability of (0.1) on a convex K of $W_0^1 L_M(\Omega)$ is proved by Gossez and Mustonen [81], more precisely they proved the existence of a solution of the following strongly nonlinear variational inequality :

$$\begin{cases} u \in K \cap \mathcal{D}(A) \\ \displaystyle\int_\Omega a(x, u, \nabla u)(\nabla u - \nabla v)\, dx + \int_\Omega g(x, u)(u - v)\, dx \\ \qquad\qquad\qquad\qquad \leq \langle f, u - v\rangle \\ \forall \;\; v \in K \cap L^\infty(\Omega) \end{cases}$$

with K satisfying certain conditions. Recently, the case of equation with g depends on x, u, and ∇u has been studied in [72] with $K = W_0^1 L_M(\Omega)$.
Another work was studied in 2004 in collaboration with M. Rhoudaf, which treat the unilateral problem associated to (0.1) (i.e. on the convex K_ψ) under the constraint of the Δ_2-condition and where the obstacle ψ is a measurable function verifying $\psi^+ \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.
Our goal in Chapter II is to study the unilateral problem associated to equation (0.1) without Δ_2-condition and regularity $\psi^+ \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.
Note that the content of this chapter is the subject of work [7] published in 2005.

I.2 Strongly nonlinear elliptic unilateral problems with natural growth in Orlicz spaces

The objective of Chapter III is to study the previous unilateral problem with L^1-dual data (a.e., $f = f_0 - \text{div}F, \quad f_0 \in L^1(\Omega), F \in (E_{\overline{M}}(\Omega)^N)$). Thus we obtained the following existence result:

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega), \ u \geq \psi \ p.p. \text{ in } \Omega, \ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f_0 T_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx, \\ v \in K_\psi \cap L^\infty(\Omega) \ \forall k > 0. \end{cases}$$

Note also that when $f \in W^{-1}E_{\overline{M}}(\Omega)$ only, an additional regularity was obtained on the solution u (i.e., $u \in K_\psi$) and it is the subject of work published [8].

I.3 Existence results for some unilateral problems without sign condition in Orlicz spaces

In this chapter we will study the unilateral problem associated to equation (0.1) with g satisfies the following growth condition,

$$|g(x,s,\xi)| \leq \gamma(x) + h(s)M(|\xi|), \tag{I.1}$$

with $\gamma \in L^1(\Omega), h \in L^1(\mathbb{R})$ and $h \geq 0$ and without any sign condition.
Several studies have been treated in the case of equation with the right hand side belongs to L^1 and where g satisfies the sign condition. We can refer for example to [50, 102] etc.
Taking $f \in L^m(\Omega)$, we found the work [60] where a bounded solution is shown for $m > N/2$ and a unbounded entropy solution is obtained for $N/2 > m > 2N/(N+2)$.
In 2002, Porretta studied the case of the equation with a second member measure.
Note that the content of this Chapter is the subject of an article published in the journal Nonlinear Analysis series A: Theory, Methods & Applications (voir [11]).

I.4 Existence of solutions for unilateral problems in L^1 involving lower order terms in divergence form in Orlicz spaces

Consider the non-linear equation:

$$(Au = -\text{div}(a(x,u,\nabla u)) - \text{div}(\phi(u)) = f, \tag{0.2}$$

where A is a partial differential operator satisfying the condition of Leray-Lions and $\phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$.

In [45] it has been proved (for $2 - \frac{1}{N} < p \leq N$), the existence and regularity of entropy solutions u of the problem (0.2) i.e.

$$\begin{cases} u \in W_0^{1,q}(\Omega) \quad \forall q < \overline{q} = \frac{N(p-1)}{N-1} \\ T_k(u) \in W_0^{1,p}(\Omega) \ \forall k > 0 \\ \displaystyle\int_\Omega a(x,u,\nabla u) T_k \nabla(u-v)\ dx + \int_\Omega \phi(u) \nabla T_k(u-v)\ dx \\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\ dx \\ \forall \ \ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{I.2}$$

In 1999, L. Boccardo and Cirmi in [46] have studied the existence and uniqueness of solution for the unilateral problem

$$\begin{cases} u \in W_0^{1,q}(\Omega) \quad \forall q < \overline{q} = \frac{N(p-1)}{N-1} \\ u \geq \psi \ p.p. \ \text{ in } \ \Omega, T_k(u) \in W_0^{1,p}(\Omega) \ \forall k > 0 \\ \displaystyle\int_\Omega a(x,\nabla u) T_k \nabla(u-v)\ dx \leq \int_\Omega f T_k(u-v)\ dx \\ \forall \ \ v \in K_\psi \cap L^\infty(\Omega), \ \ K_\psi = \Big\{ u \in W_0^{1,p}(\Omega)/u \geq \psi \ p.p. \ \text{ in } \ \Omega \Big\} \end{cases} \tag{I.3}$$

with $f \in L^1(\Omega), \psi : \Omega \to I\!\!R_+$ is a measurable function such that $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Note that in parallel, there is a development of resolution of the problems (I. 2) and (I, 3) where a satisfies the more general growth condition, which requires the resolution in the context of Orlicz-Sobolev spaces. In this setting is proved by Gossez and Mustonen [81] the existence of solution of the following inequality variational :

$$\begin{cases} u \in K \cap \mathcal{D}(A) \\ \displaystyle\int_\Omega a(x,u,\nabla u)(\nabla u - \nabla v)\ dx \leq \langle f, u-v \rangle \\ \forall \ \ v \in K \end{cases}$$

with K satisfying certain conditions.
In the particular case where K relative to an obstacle ψ, i.e.,

$$K_\psi = \Big\{ u \in W_0^1 L_M(\Omega)/u \geq \psi \ p.p. \ \text{ in } \ \Omega \Big\}$$

we found the work of Benkirane and Bennouna [38], under a constraint on the N-function, to verify the Δ_2-condition. These last two authors have treated the case $\phi \neq 0$ with $K = W_0^1 L_M(\Omega)$ (see [36]).
In this Chapter, we shall be concerned with the existence result of unilateral problem associated to the equation (0.2) with the second member $f \in L^1(\Omega)$ and without the Δ_2-condition and where the obstacle ψ is only measurable.
We first studied (see Theorem 5.2.1), the case where $a \equiv a(x,\xi)$ satisfies the following coercivity condition

$$a(x,\xi)(\xi - \nabla v_0) \geq \alpha M(|\xi|) - \delta(x)$$

and next (see Theorem 5.2.2), the case when $a \equiv a(x,s,\xi)$, with

$$a(x,\xi).\xi \geq \alpha M(\frac{|\xi|}{\lambda}).$$

The results of this Chapter are the subject of two articles published (see [12, 13]).

Chapter 1

PRELIMINARIES

In this Chapter we will give the ingredients necessary in we will serve later.

1.1. Orlicz-Sobolev spaces

1.1.1. $N-$functions

A function $M : I\!R \to I\!R$ is said to be an $N-$function, if and only if

1) M is even, continuous and convex,

2) $M(t) = 0$ if and only if $t = 0$,

3) $\lim\limits_{t \to 0} M(t)/t = 0$, $\lim\limits_{t \to \infty} M(t)/t = \infty$.

A function $M : I\!R \to I\!R$ is an $N-$function, if and only if it can be represented as an integral (see [86])

$$M(t) = \int_0^{|t|} m(s)\, ds,$$

where $m : [0, \infty[\to [0, \infty[$ is increasing, right-continuous, $m(t) = 0$ if and only if $t = 0$, and $\lim\limits_{t \to \infty} m(t) = \infty$. A conjugate $N-$function $\overline{M}$ for an $N-$function $M(t) = \int_0^{|t|} m(s)\, ds$, is defined by

$$\overline{M}(t) = \int_0^{|t|} \overline{m}(s)\, ds,$$

where

$$\overline{m}(t) = \sup_{m(s) \leq t} s$$

is the right inverse of m. From the definition of $\overline{m}$ it follows that

$$\overline{m}(m(t)) \geq t \;\text{ and }\; m(\overline{m}(s)) \geq s \;\text{ for all }\; t, s \geq 0. \tag{1.1.1}$$

If the function m is continuous, we have equality in (1.1.1), which means that m and $\overline{m}$ are mutual inverses.
Next we present some basic inequalities connected with $N-$functions (see [86]).

Lemma 1.1.1. *Let* $M(t) = \int_0^{|t|} m(s)\, ds$, *be an* $N-$*function. Then*

1) *(Young's inequality)* $ts \leq M(t) + \overline{M}(s)$ *for all* $t, s \geq 0$ *and equality holds if and only if* $t = \overline{m}(s)$ *or* $s = m(t)$,

2) $t \leq M^{-1}(t)\overline{M}^{-1}(t) \leq 2t$ *for all* $t \geq 0$,

3) $\overline{M}(\frac{M(t)}{t}) \leq M(t)$ *for all* $t > 0$.

Definition 1.1.1. *An* $N-$*function* M, *satisfies the* Δ_2-*condition denote* $M \in \Delta_2$, *if there exists constants* $k > 0$ *such that*

$$M(2t) \leq kM(t) \quad \textit{for all} \quad t \geq 0 \tag{1.1.2}$$

it is readily seen that this will be the case if and only if for every $r > 1$ *there exists a positive constant* $k = k(r)$ *such that for all* $t \geq 0$

$$M(rt) \leq kM(t) \quad \textit{for all} \quad t \geq 0. \tag{1.1.3}$$

Let P and Q be two $N-$functions. Q dominate P denote $P \prec Q$, if there exist $k > 0$ such that:

$$P(t) \leq Q(kt), \quad \forall \ t \geq 0. \tag{1.1.4}$$

Similarly, Q dominate P near infinity if there exist $k > 0$ and $t_0 > 0$ such that (1.1.4) hold only for $t \geq t_0$. In this case there exist $K > 0$ such that:

$$P(t) \leq Q(kt) + K, \quad \forall \ t \geq 0.$$

We shall say that the $N-$functions P and Q are equivalent and write $P \sim Q$ if $P \prec Q$ and $Q \prec P$.
It follows from the definition that the $N-$functions P and Q are equivalent if, and only if, there exist a positive constants k_1, k_2 and t_0 such that

$$P(k_1 t) \leq Q(t) \leq P(k_2 t), \quad \forall \ t \geq t_0. \tag{1.1.5}$$

We say that P increases essentially more slowly than Q near infinity, denote $P \ll Q$, if for every $k > 0$ $\lim_{t\to\infty} \frac{P(kt)}{Q(t)} = 0$. This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

1.1.2. Orlicz spaces

Let Ω be an open subset of $I\!R^N$. The Orlicz class $\mathcal{K}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\, dx < +\infty \quad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\, dx < +\infty \text{ for some } \lambda > 0 \right).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0, \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\, dx \leq 1 \right\}$$

and $\mathcal{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space.
The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\, dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.
Let X and Y be arbitrary Banach spaces with bilinear bicontinuous pairing $\langle , \rangle_{X,Y}$.
We say that a sequence $\{u_n\} \subset X$ converges to $u \in X$ with respect to the topology $\sigma(X, Y)$, denote $u_n \to u \;\; (\sigma(X, Y))$ in X, if $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in Y$. For exemple, if $X = L_M(\Omega)$ and $Y = L_{\overline{M}}(\Omega)$, then the pairing is defined by $\langle u, v\rangle = \int_\Omega u(x)v(x)\, dx$ for all $u \in X, v \in Y$.

Theorem 1.1.1 *[109]. Let M be an $N-$function and $\Omega \subset I\!R^N$ open and bounded then.*

1) $E_M(\Omega) \subset \mathcal{K}_M(\Omega) \subset L_M(\Omega)$,

2) $E_M(\Omega) = L_M(\Omega)$ if and only if $M \in \Delta_2$,

3) $E_M(\Omega)$ is separable,

4) $L_M(\Omega)$ is reflexive if and only if $M \in \Delta_2$ and $\overline{M} \in \Delta_2$.

1.1.3. Orlicz-Sobolev spaces

We define Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives of order 1 belongs in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.
The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
We recall that a sequence u_n in $L_M(\Omega)$ is said to be convergent to $u \in L_M(\Omega)$ modular, denote $u_n \to u$ (mod) in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M(\frac{|u_n(x) - u(x)|}{\lambda})\, dx \to 0 \;\; \text{as} \;\; n \to +\infty.$$

This implies that u_n converges to u for $\sigma(L_M(\Omega), L_{\overline{M}}(\Omega))$. A similar definition can be given in $W^1L_M(\Omega)$ where one requires the above for u and each of its first derivatives. If M satisfies the Δ_2-condition (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.
Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).
In the sequel we will need the following Lemmas (see Lemma 4.4 and 5.7 of [77])

Lemma 1.1.2. *Let $(u_n)_n \subset L_M(\Omega)$ a bounded sequence in $L_M(\Omega)$ such that $u_n \to u$ a.e. in Ω. Then, $u \in L_M(\Omega)$ and $u_n \rightharpoonup u$ weakly in $L_M(\Omega)$ for $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 1.1.3. *Let Ω be an open bounded of $I\!R^N$. Then there exists two positive contants c_1 and c_2 such that*

$$\int_\Omega M(c_1|v|)\, dx \leq c_2 \int_\Omega M(|\nabla v|)\, dx$$

for all $v \in W_0^1L_M(\Omega)$.

We recall some Lemmas introduced in [39] which will be used later.

Lemma 1.1.4. *Let $F : I\!R \to I\!R$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i}u & a.e.\ in \quad \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e.\ in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 1.1.5. *Let $F : I\!R \to I\!R$ be uniformly Lipschitzian, with $F(0) = 0$. We assume that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $T_F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ defined by $T_F(u) = F(u)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

We have already used the concept of a Carathéodory function. Therefore, let us only recall that a function $g = g(x, s)$, defined for $x \in \Omega$ and $s \in I\!R^m$, is called a Carathéodory function, if

1) the function $g(x, .)$ is continuous for a.e. $x \in \Omega$,

2) the function $g(., s)$ is measurable on Ω for every $s \in I\!R^m$.

For every Carathéodory function g, we define the Nemytskii operator G generated by g and acting on vector-valued functions $u = u(x), u : \Omega \to I\!R^m$, by the formula

$$(Gu)(x) = g(x, u(x)), \quad x \in \Omega.$$

It is well known that the operator G maps the space $\prod_{i=1}^{m} L^{p_i}(\Omega)$ continuously into $L^p(\Omega)$ if and only if the following estimate holds:

$$|g(x,s)| \leq a(x) + c\sum_{i=1}^{m} |s|^{\frac{p_i}{p}}$$

for a.e. $x \in \Omega$ and every $s \in I\!R^m$ with a fixed (nonnegative) function $a \in L^p(\Omega)$ and a fixed nonnegative constant c.
We give now the following Lemma which concerns operators of Nemytskii type in Orlicz spaces (see [39]).

Lemma 1.1.6. *Let Ω be an open subset of $I\!R^N$ with finite measure. Let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times I\!R \to I\!R$ be a Carathéodory function such that, for* a.e. $x \in \Omega$ and all $s \in I\!R$:

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.
Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We introducing the truncature operator. For a given constant $k > 0$ we define the cut function $T_k : I\!R \to I\!R$ as

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k \\ k \ \text{sign}(s) & \text{if} \quad |s| > k. \end{cases} \tag{1.1.6}$$

For a function $u = u(x), x \in \Omega$, we define the truncated function $T_k u = T_k(u)$ pointwise: for every $x \in \Omega$ the value of $(T_k u)$ at x is just $T_k(u(x))$.
We now introduce the functional spaces we will need later.
For an $N-$function M, $\mathcal{T}_0^{1,M}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to I\!R$ such that for all $k > 0$ the truncated function $T_k(u) \in W_0^1 L_M(\Omega)$.
We gives the following Lemma this is a generalization of Lemma 2.1 [34] in Orlicz spaces.

Lemma 1.1.7. *For every $u \in \mathcal{T}_0^{1,M}(\Omega)$, there exists a unique measurable function $v : \Omega \longrightarrow I\!R^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \text{almost everywhere in } \Omega, \quad \text{for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 1.1.8. *Let $\lambda \in I\!R$ and let u and v be two functions which are finite almost everywhere, and which belongs to $\mathcal{T}_0^{1,M}(\Omega)$. Then,*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \ \text{a.e. in } \Omega,$$

where ∇u, ∇v and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in Lemma 1.1.7.

The proof of this Lemma is similar to the proof of Lemma 2.12 [66] for the L^p case . Below, we will use the following technicals Lemmas.

Lemma 1.1.9. *Let $(f_n)_n, f \in L^1(\Omega)$ such that*

1) $f_n \geq 0$ a.e. in Ω

2) $f_n \to f$ a.e. in Ω

3) $\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$. Then $f_n \to f$ strongly in $L^1(\Omega)$.

We say that $\Omega \subset I\!R^N$ satisfies the segment property if there exists a locally finite open covering $\{\mathcal{O}\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\}$ such that for $x \in \overline{\Omega} \cap \mathcal{O}$ and $0 < t < 1$, one $x + ty_i \in \Omega$.

Lemma 1.1.10 *[36]. Let Ω be an open bounded subset of $I\!R^N$ satisfies the segment property. If $u \in W_0^1 L_M(\Omega)$, then*

$$\int_\Omega \operatorname{div}\, u\, dx = 0.$$

1.1.4. Imbedding Theorem in Orlicz-Sobolev spaces

Let Ω be an open subset of $I\!R^N$, with the segment property. Let M an $N-$function and we suppose that

$$\int_0^\infty \frac{M^{-1}(s)}{s^{1+\frac{1}{N}}}\,ds = +\infty.$$

Let $\tilde{M}$ an $N-$funcion equal to M near to infinity such that

$$\int_0^1 \frac{\tilde{M}^{-1}(s)}{s^{1+\frac{1}{N}}}\,ds < +\infty$$

(see [2] for the construction of this $N-$function).
We define a new $N-$function $\tilde{M}_1$ by

$$\tilde{M}_1(t) = \int_0^t \frac{M^{-1}(s)}{s^{1+\frac{1}{N}}}\,ds$$

and let M_1 an $N-$function equal to M near 0 and $\tilde{M}_1$ to infinity.
Repeating this process we obtain a finite sequence of $N-$functions: $M_1, M_2 = (M_1)_1, ..., M_q$ where $q = q(M, N)$ such that

$$\int_1^\infty \frac{M_{q-1}^{-1}(s)}{s^{1+\frac{1}{N}}}\,ds = +\infty \quad \text{but} \quad \int_1^\infty \frac{M_q^{-1}(s)}{s^{1+\frac{1}{N}}}\,ds < +\infty$$

if

$$\int_1^\infty \frac{M_q^{-1}(s)}{s^{1+\frac{1}{N}}}\,ds < +\infty$$

we put $q(M,N)=0$.
We say that Ω has the cone property if each of its points is the vertex of a finite right-spherical cone contained in Ω and congruent to some finite right-spherical cone.
We adopt the following imbedding Theorem from [65]

Lemma 1.1.11. *Let Ω be an open subset of $I\!R^N$ with cone property.*
If $m \leq q(M,N)$ then $W^1L_M(\Omega) \subset L_{M_m}$ with continuous imbedding.
If $m > q(M,N)$ then $W^1L_M(\Omega) \subset C(\Omega) \cap L^\infty(\Omega)$ with compact imbedding.

So, in the two case, there exists an N-function Q such that $M \ll Q$ and $W^1L_M(\Omega) \subset L_Q$ (see [1]).
It's known that, there exists also an N-function Q such that

$$M \ll Q \text{ and } W^1L_M(\Omega) \subset E_Q(\Omega)$$

with compact imbedding ([1, 65]).
In particular, we get $W^1L_M(\Omega) \subset E_M(\Omega)$ with compact imbedding.
When Ω is an any open subset of $I\!R^N$, the imbedding of the previous Lemma remain valid for $W_0^1L_M(\Omega)$ instead of $W^1L_M(\Omega)$.
Then, we deduce also that there exists an N-function Q such that

$$M \ll Q \text{ and } W^1L_M(\Omega) \subset E_Q(\Omega)$$

with continuous imbedding (which can be supposed compact). Moreover, we have

$$W^1L_M(\Omega) \subset E_M(\Omega) \text{ with compact imbedding.}$$

An other applications of the previous Lemma allows to have for any subset Ω,

$$M \ll Q \text{ and } W^1L_M(\Omega) \subset E_Q^{\text{loc}}(\Omega),$$

for some N-function Q (and also the compact imbedding).
Thus, $W^1L_M(\Omega) \subset E_M^{\text{loc}}(\Omega)$ with continuous and compact imbedding.

1.2. Notations

In the sequel, we use the following notations
Denoting by $\epsilon(n,j,h)$ any quantity such that

$$\lim_{h\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,h)=0.$$

If the quantity we consider does not depend on one parameter among n, j and h, we will omit the dependence on the corresponding parameter: as an example, $\epsilon(n,h)$ is any quantity such that

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\epsilon(n,h)=0.$$

Finally, we will denote (for example) by $\epsilon_h(n,j)$ a quantity that depends on n, j, h and is such that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon_h(n,j)=0$$

for any fixed value of h.

Chapter 2

EXISTENCE OF SOLUTIONS FOR UNILATERAL PROBLEMS IN L^1 IN ORLICZ SPACES [1]

In this Chapter, we shall be concerned with the existence result of Unilateral problem associated to the equations of the form,

$$Au + g(x, u, \nabla u) = f,$$

where A is a Leray-Lions operator from its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1}E_{\overline{M}}(\Omega)$. On the nonlinear lower order term $g(x, u, \nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, and satisfies the sign condition. The right hand side f belongs to $L^1(\Omega)$.

2.1. Introduction

Let Ω be an open bounded subset of $I\!R^N, N \geq 2$, with segment property. Let us consider the following nonlinear Dirichlet problem

$$-\text{div}(a(x, u, \nabla u)) + g(x, u, \nabla u) = f \tag{2.1.1}$$

where $f \in L^1(\Omega), Au = -\text{div}a(x, u, \nabla u)$ is a Leray-Lions operator defined on its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$, with M an N-function and where g is a nonlinearity with the "natural" growth condition:

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|))$$

and which satisfies the classical sign condition

$$g(x, s, \xi).s \geq 0.$$

In the case where $f \in W^{-1}E_{\overline{M}}(\Omega)$, an existence Theorem has been proved in [79] with the nonlinearities g depends only on x and u, and in [39] where g depends also the ∇u.

[1] Journal of inequality in pur and applied mathematics, vol. 6, Issue 2, Article 54, 2005.

For the case where $f \in L^1(\Omega)$, the author's in [40] studied the problem (2.1.1), with the following exact natural growth

$$|g(x,s,\xi)| \geq \beta M(|\xi|) \text{ for } |s| \geq \mu$$

and in [43] no coercivity condition is assumed on g but the result is restricted to N-function M satisfying a Δ_2-condition, while in [73] the author's were concerned of the above problem without assuming a Δ_2-condition on M.
The purpose of this Chapter is to prove an existence result for unilateral problems associated to (2.1.1) whithout assuming the Δ_2-condition in the setting of Orlicz-Sobolev space.
Further work for equation (2.1.1) in the L^p case can be found in [102] , and in [74] in the case of obstacle problems.

2.2. Main results

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$, with the segment property.
Given an obstacle $\psi : \Omega \to \overline{I\!R}$ which is a measurable function and consider the set

$$K_\psi = \{u \in W_0^1 L_M(\Omega);\ u \geq \psi \ \ a.e. \text{ in } \ \Omega\}. \tag{2.2.1}$$

We now state our hypotheses on the differential operator A defined by,

$$Au = -\text{div}(a(x,u,\nabla u)). \tag{2.2.2}$$

(A_1) $a(x,s,\xi) : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function.

(A_2) There exist tow N-function M and P with $P \ll M$, function $k(x)$ in $E_{\overline{M}}(\Omega)$, positive constants k_1, k_2, k_3, k_4 such that,

$$|a(x,s,\zeta)| \leq k(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|),$$

for a.e. x in Ω and for all $s \in I\!R, \zeta \in I\!R^N$.

(A_3) For a.e. x in Ω, $s \in I\!R$ and ζ, ζ' in $I\!R^N$, with $\zeta' \neq \zeta$

$$[a(x,s,\zeta) - a(x,s,\zeta')](\zeta - \zeta') > 0$$

(A_4) There exist $\delta(x)$ in $L^1(\Omega)$, strictly positive constant α such that, for some fixed element v_0 in $K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$.

$$a(x,s,\zeta)(\zeta - \nabla v_0) \geq \alpha M(|\zeta|) - \delta(x)$$

for a.e. x in Ω, and all $s \in I\!R$, $\zeta \in I\!R^N$.

(A_5) For each $v \in K_\psi \cap L^\infty(\Omega)$ there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that,

$$v_j \to v \quad \text{for the modular convergence.}$$

Furthermore, let $g : \Omega \times I\!R \times I\!R^N \to I\!R$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in I\!R, \zeta \in I\!R^N$

(G_1) $g(x, s, \zeta)s \geq 0$,

(G_2) $|g(x, s, \zeta)| \leq b(|s|)\,(c(x) + M(|\zeta|))$,
where $b : I\!R_+ \to I\!R_+$ is continuous non decreasing function, and c is a given nonegative function in $L^1(\Omega)$.

Remark 2.2.1. *Remark that the condition (A_5) holds if the one of the following conditions is verified:*

a) There exists $\overline{\psi} \in K_\psi$ such that $\psi - \overline{\psi}$ is continuous in Ω, (see [[81], Proposition 9]),

b) $\psi \in W_0^1 E_M(\Omega)$, (see [[81], Proposition 10]),

c) The N-function M satisfies the Δ_2-condition,

d) $\psi = -\infty$ (i.e. $K_\psi = W_0^1 L_M(\Omega)$) (see Remark 2 of [81] and Theorem 4 of [80]).

Let us recall the following Lemma which will be needed later :

Lemma 2.2.1 *[77]. Let $f \in W^{-1}E_{\overline{M}}(\Omega)$ and let $K \subset W_0^1 L_M(\Omega)$ be convex, $\sigma(\prod L_M, \prod E_{\overline{M}})$ sequentially closed and such that $K \cap W_0^1 E_M(\Omega)$ is $\sigma(\prod L_M, \prod L_{\overline{M}})$ dense in K. Assume that (A_1)-(A_4) are satisfy with $v_0 \in K \cap W_0^1 E_M(\Omega)$, then the variational inequality*

$$\begin{cases} u \in \mathcal{D}(A) \cap K, \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla(u - v)\, dx \leq \langle f, u - v\rangle, \\ \forall \quad v \in K. \end{cases}$$

has at least one solution.

Remark 2.2.2. *The previous Lemma can be applied if $K = W_0^1 L_M(\Omega)$ (see Remark 2.2.1 and Remark 2 of [81]).*

Remark 2.2.3. *Remark that the convex set K_ψ satisfies the following conditions :*

1) K_ψ is $\sigma(\prod L_M, \prod E_{\overline{M}})$ sequentially closed.

2) $K_\psi \cap W_0^1 E_M(\Omega)$ is $\sigma(\prod L_M, \prod L_{\overline{M}})$ dense in K_ψ.

Proof **:**

1) Let $u_n \in K_\psi$ which converges to $u \in W_0^1 L_M(\Omega)$ for $\sigma(\prod L_M, \prod E_{\overline{M}})$. Since the imbedding of $W_0^1 L_M(\Omega)$ into $E_M(\Omega)$ is compact it follows that for a subsequence $u_n \to u$ a.e. in Ω, which gives $u \in K_\psi$.

2) It suffices to apply (A_5) and the fact that $T_n(u) \to u$ (mod) in $W^1 L_M(\Omega)$ for all $u \in K_\psi$.

We shall prove the following existence Theorem.

Theorem 2.2.1. *Let $f \in L^1(\Omega)$. Assume that (A_1) - (A_5), (G_1) and (G_2) hold, then there exists at least one solution of the following unilateral problem,*

$$(P)\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega),\ u \geq \psi \text{ a.e. in } \Omega,\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, \\ \forall\ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}$$

2.3. Proof of Theorem 2.2.1

Let us give and prove the following Lemma which are needed below.

Lemma 2.3.1. *Assume that $(A_1) - (A_4)$ are satisfied, and let $(z_n)_n$ be a sequence in $W_0^1 L_M(\Omega)$ such that*

a) $z_n \rightharpoonup z$ in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$;

b) $(a(x,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$;

c) $\int_\Omega [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx \to 0$ as n and $s \to +\infty$. (Where χ_s the characteristic function of $\Omega_s = \{x \in \Omega, |\nabla z| \leq s\}$).

Then

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad in \quad L^1(\Omega).$$

Remark 2.3.1. *The condition b) thoes not necessary in the case where the N-function M satisfies the Δ_2-condition.*

Proof **:**
The condition a) implies that the sequence $(z_n)_n$ is bounded in $W_0^1 L_M(\Omega)$, hence there exists two positive constants λ, C such that

$$\int_\Omega M(\lambda|\nabla z_n|)\,dx \leq C. \tag{2.3.1}$$

On the other hand, let Q be an N-function such that $M \ll Q$ and the continuous imbedding $W_0^1 L_M(\Omega) \subset E_Q(\Omega)$ holds (see [77]). Let $\varepsilon > 0$. Then there exists $C_\varepsilon > 0$, as in [39], such that

$$|a(x,s,\zeta)| \leq c(x) + C_\varepsilon + k_1 \overline{M}^{-1} Q(\varepsilon|s|) + k_3 \overline{M}^{-1} M(\varepsilon|\zeta|) \tag{2.3.2}$$

for a.e. $x \in \Omega$ and for all $(s, \zeta) \in I\!R \times I\!R^N$. From (2.3.1) and (2.3.2) we deduce that $(a(x, z_n, \nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

Proof of Lemma 2.3.1

Fix $r > 0$ and let $s > r$ since $\Omega_r \subset \Omega_s$ we have,

$$\begin{aligned} 0 &\leq \int_{\Omega_r} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)][\nabla z_n - \nabla z] \, dx \\ &\leq \int_{\Omega_s} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)][\nabla z_n - \nabla z] \, dx \\ &= \int_{\Omega_s} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s] \, dx \\ &\leq \int_{\Omega} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s] \, dx. \end{aligned} \tag{2.3.3}$$

Which with the condition $c)$ imply that,

$$\lim_{n\to\infty} \int_{\Omega_r} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)][\nabla z_n - \nabla z] \, dx = 0. \tag{2.3.4}$$

So, as in [77]

$$\nabla z_n \to \nabla z \quad a.e. \text{ in } \Omega. \tag{2.3.5}$$

On the one side, we have

$$\begin{aligned} \int_{\Omega} a(x, z_n, \nabla z_n)\nabla z_n \, dx = \int_{\Omega} &[a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)] \\ &\times[\nabla z_n - \nabla z\chi_s] \, dx \\ &+ \int_{\Omega} a(x, z_n, \nabla z\chi_s)(\nabla z_n - \nabla z\chi_s) \, dx \\ &+ \int_{\Omega} a(x, z_n, \nabla z_n)\nabla z\chi_s \, dx. \end{aligned} \tag{2.3.6}$$

Since $(a(x, z_n, \nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, from (2.3.5), we obtain

$$a(x, z_n, \nabla z_n) \rightharpoonup a(x, z, \nabla z) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{2.3.7}$$

Consequently,

$$\int_{\Omega} a(x, z_n, \nabla z_n)\nabla z\chi_s \, dx \to \int_{\Omega} a(x, z, \nabla z)\nabla z\chi_s \, dx \tag{2.3.8}$$

as $n \to \infty$.

letting also $s \to \infty$, we obtain

$$\int_{\Omega} a(x, z, \nabla z)\nabla z\chi_s \, dx \to \int_{\Omega} a(x, z, \nabla z)\nabla z \, dx. \tag{2.3.9}$$

On the other side, it is easy to see that the second term of the right hand side of (2.3.6) tend to 0 as $n \to \infty$ and $s \to \infty$.

Moreover, from $c)$, (2.3.8) and (2.3.9) we have,

$$\lim_{n\to\infty} \int_{\Omega} a(x, z_n, \nabla z_n)\nabla z_n \, dx = \int_{\Omega} a(x, z, \nabla z)\nabla z \, dx, \tag{2.3.10}$$

hence

$$\lim_{n\to\infty}\int_\Omega a(x,z_n,\nabla z_n)(\nabla z_n-\nabla v_0)\,dx=\int_\Omega a(x,z,\nabla z)(\nabla z-\nabla v_0)\,dx.$$

Finally, using (A_4), one obtains, by Lemma 1.1.9 and Vitali's Theorem,

$$M(|\nabla z_n|)\longrightarrow M(|\nabla z|)\ \text{ in }\ L^1(\Omega).$$

To prove the existence Theorem, we proceed by steps.

STEP 1. Approximate problems.

Let us define

$$g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$$

and let us consider the approximate problems:

$$(P_n)\begin{cases} u_n\in K_\psi\cap\mathcal{D}(A),\\ \langle Au_n,u_n-v\rangle+\displaystyle\int_\Omega g_n(x,u_n,\nabla u_n)(u_n-v)\,dx\le\int_\Omega f_n(u_n-v)\,dx,\\ \forall v\in K_\psi. \end{cases}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. Applying Lemma 2.2.1 the problem (P_n) has at least one solution.

STEP 2. A priori estimates.

Let $k\ge\|v_0\|_\infty$ and let $\varphi_k(s)=se^{\gamma s^2}$, where $\gamma=(\frac{b(k)}{\alpha})^2$.
It is well know that

$$\varphi_k'(s)-\frac{b(k)}{\alpha}|\varphi_k(s)|\ge\frac{1}{2},\quad\forall s\in I\!R. \tag{2.3.11}$$

Taking $u_n-\eta\varphi_k(T_l(u_n-v_0))$ as test function in (P_n), where $l=k+\|v_0\|_\infty$, we obtain,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_l(u_n-v_0)\varphi_k'(T_l(u_n-v_0))\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n-v_0))\,dx$$
$$\le\int_\Omega f_n\varphi_k(T_l(u_n-v_0))\,dx.$$

Since $g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n-v_0))\ge0$ on the subset $\{x\in\Omega:|u_n(x)|>k\}$, then

$$\int_{\{|u_n-v_0|\le l\}}a(x,u_n,\nabla u_n)\nabla(u_n-v_0)\varphi_k'(T_l(u_n-v_0))\,dx$$
$$\le\int_{\{|u_n|\le k\}}|g_n(x,u_n,\nabla u_n)||\varphi_k(T_l(u_n-v_0))|\,dx+\int_\Omega f_n\varphi_k(T_l(u_n-v_0))\,dx.$$

by using (A_4) and (G_1), we have

$$\alpha\int_{\{|u_n-v_0|\le l\}}M(|\nabla u_n|)\varphi_k'(T_l(u_n-v_0))\,dx$$
$$\le b(k)\int_\Omega(c(x)+M(|\nabla T_k(u_n)|))\,|\varphi_k(T_l(u_n-v_0))|\,dx$$
$$+\int_\Omega\delta(x)\varphi_k'(T_l(u_n-v_0))\,dx+\int_\Omega f_n\varphi_k(T_l(u_n-v_0))\,dx.$$

Since

$$\{x \in \Omega, |u_n(x)| \leq k\} \subseteq \{x \in \Omega : |u_n - v_0| \leq l\}$$

and the fact that $c, \delta \in L^1(\Omega)$, further f_n is bounded in $L^1(\Omega)$, then

$$\int_\Omega M(|\nabla T_k(u_n)|)\varphi_k'(T_l(u_n - v_0))\,dx \leq \frac{b(k)}{\alpha}\int_\Omega M(|\nabla T_k(u_n)|)|\varphi_k(T_l(u_n - v_0))|\,dx + c_k$$

where c_k is a positive constant depend of the k. Which implies that

$$\int_\Omega M(|\nabla T_k(u_n)|)\left[\varphi_k'(T_l(u_n - v_0)) - \frac{b(k)}{\alpha}|\varphi_k(T_l(u_n - v_0))|\right]\,dx \leq c_k.$$

By using (2.3.11), we deduce,

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx \leq 2c_k. \tag{2.3.12}$$

Since $T_k(u_n)$ is bounded in $W_0^1 L_M(\Omega)$, there exists some $v_k \in W_0^1 L_M(\Omega)$ such that

$$\begin{array}{l} T_k(u_n) \rightharpoonup v_k \ \text{ weakly in } \ W_0^1 L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M, \Pi E_{\overline{M}}) \\ T_k(u_n) \to v_k \text{ strongly in } \ E_M(\Omega). \end{array} \tag{2.3.13}$$

STEP 3. Convergence in measure of u_n

Let $k_0 \geq \|v_0\|_\infty$ and $k > k_0$, taking $v = u_n - T_k(u_n - v_0)$ as test function in (P_n) gives,

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)T_k(u_n - v_0)\,dx \\ \leq \int_\Omega f_n T_k(u_n - v_0)\,dx,\end{aligned} \tag{2.3.14}$$

since $g_n(x, u_n, \nabla u_n)T_k(u_n - v_0) \geq 0$ on the subset $\{x \in \Omega, |u_n(x)| > k_0\}$ hence (2.3.14) implies that,

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\,dx \leq k\int_{\{|u_n| \leq k_0\}} |g_n(x, u_n, \nabla u_n)|\,dx + k\|f\|_{L^1(\Omega)}$$

which gives, by using (G_1),

$$\begin{aligned}&\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\,dx \\ &\qquad \leq kb(k_0)\left[\int_\Omega |c(x)|\,dx + \int_\Omega M(|\nabla T_{k_0}(u_n)|)\,dx\right] + kc.\end{aligned} \tag{2.3.15}$$

Combining (2.3.12) and (2.3.15), we have,

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\,dx \leq k[c_{k_0} + c].$$

By (A_4), we obtain,

$$\int_{\{|u_n - v_0| \leq k\}} M(|\nabla u_n|)\,dx \leq kc_1$$

where c_1 independent of k, since k arbitrary, we have

$$\int_{\{|u_n|\leq k\}} M(|\nabla u_n|)\,dx \leq \int_{\{|u_n-v_0|\leq k+\|v_0\|_\infty\}} M(|\nabla u_n|)\,dx \leq kc_2$$

i.e.,

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx \leq kc_2. \tag{2.3.16}$$

Now, we prove that u_n converges to some function u in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We shall show that u_n is a Cauchy sequence in measure.
Let $k > 0$ large enough, thanks to Lemma 1.1.3 of Chapter I, there exists two positive constants c_3 and c_4 such that

$$\int_\Omega M(c_3T_k(u_n))\,dx \leq c_4\int_\Omega M(|\nabla T_k(u_n)|)\,dx \leq kc_5, \tag{2.3.17}$$

then, we deduce, by using (2.3.17) that

$$M(c_3k) \ \text{ meas } \ \{|u_n| > k\} = \int_{\{|u_n|>k\}} M(c_3T_k(u_n))\,dx \leq c_5k,$$

hence

$$\text{meas}(|u_n| > k) \leq \frac{c_5k}{M(c_3k)} \quad \forall n, \forall k. \tag{2.3.18}$$

Letting k to infinity, we deduce that, meas$\{|u_n| > k\}$ tend to 0 as k tend to infinity.
For every $\lambda > 0$, we have

$$\begin{aligned}\text{meas}(\{|u_n - u_m| > \lambda\}) \leq \text{meas}(\{|u_n| > k\}) + &\ \text{meas}(\{|u_m| > k\})\\ +\text{meas}(\{|T_k(u_n) - T_k(u_m)| &> \lambda\}).\end{aligned} \tag{2.3.19}$$

Consequently, by (2.3.13) we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.
Let $\epsilon > 0$ then, by (2.3.19) there exists some $k(\epsilon) > 0$ such that meas$(\{|u_n - u_m| > \lambda\}) < \epsilon$ for all $n, m \geq h_0(k(\epsilon), \lambda)$. This proves that $(u_n)_n$ is a Cauchy sequence in measure in Ω , thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned}&T_k(u_n) \rightharpoonup T_k(u) \ \text{ weakly in } \ W_0^1L_M(\Omega) \ \text{ for } \ \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}),\\ &T_k(u_n) \to T_k(u) \ \text{ strongly in } \ E_M(\Omega) \ \text{ and a.e. in } \ \Omega.\end{aligned} \tag{2.3.20}$$

Step 4. Boundedeness of $(a(x, T_k(u_n), \nabla T_k(u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$.

Let $w \in (E_M(\Omega))^N$ be arbitrary, by (A_3) we have

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) \geq 0,$$

which implies that

$$a(x, u_n, \nabla u_n)(w - \nabla v_0) \leq a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0) - a(x, u_n, w)(\nabla u_n - w)$$

and integrating on the subset $\{x \in \Omega, \quad |u_n - v_0| \leq k\}$, we obtain,

$$\int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(w-\nabla v_0)\,dx \leq \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx + \int_{\{|u_n-v_0|\leq k\}} a(x,u_n,w)(w-\nabla u_n)\,dx. \tag{2.3.21}$$

We claim that ,

$$\int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n - v_0)\,dx \leq c_{10}, \tag{2.3.22}$$

with c_{10} is a positive constant depending of k.
Indeed, if we take $v = u_n - T_k(u_n - v_0)$ as test function in (P_n), we get,

$$\int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v_0)\,dx \leq \int_\Omega f_n T_k(u_n-v_0)\,dx.$$

Since $g_n(x,u_n,\nabla u_n)T_k(u_n - v_0) \geq 0$ on the subset $\{x \in \Omega, \quad |u_n| \geq \|v_0\|_\infty\}$, which implies

$$\int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx \leq b(\|v_0\|_\infty)\int_\Omega (c(x) + M(\nabla T_{\|v_0\|_\infty}(u_n))\,dx + k\|f\|_{L^1(\Omega)}. \tag{2.3.23}$$

Combining (2.3.12) and (2.3.23), we deduce (2.3.22).
On the other hand, for λ large enough, we have by using (A_2)

$$\int_{\{|u_n-v_0|\leq k\}} \overline{M}(\frac{a(x,u_n,w)}{\lambda})\,dx \leq \int_\Omega \overline{M}(\frac{k(x)}{\lambda}) + \frac{k_3}{\lambda}\int_\Omega M(k_4|w|) + c \leq c_{11}, \tag{2.3.24}$$

hence, $|a(x,u_n,w)|\chi_{\{|u_n-v_0|\leq k\}}$ bounded in $L_{\overline{M}}(\Omega)$. Which implies that the second term of the right hand side of (2.3.21) is bounded.
Consequently, we obtain,

$$\int_{\{|u_n-v_0|\leq k\}} a(x,u_n,\nabla u_n)(w-\nabla v_0)\,dx \leq c_{12}, \tag{2.3.25}$$

with c_{12} is a positive constant depending of k .
Hence, by the Theorem of Banach-Steinhaus, the sequence $(a(x,u_n,\nabla u_n))\chi_{\{|u_n-v_0|\leq k\}})_n$ remains bounded in $(L_{\overline{M}}(\Omega))^N$. Since k arbitrary, we deduce that $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is also bounded in $(L_{\overline{M}}(\Omega))^N$. Which implies that, for all $k > 0$ there exists a function $h_k \in (L_{\overline{M}}(\Omega))^N$, such that,

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in} \quad (L_{\overline{M}}(\Omega))^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega)). \tag{2.3.26}$$

STEP 5. Almost everywhere convergence of the gradient.

We fix $k > \|v_0\|_\infty$. Let $\Omega_s = \{x \in \Omega, |\nabla T_k(u(x))| \leq s\}$ and denote by χ_s the characteristic function of Ω_s. Clearly, $\Omega_s \subset \Omega_{s+1}$ and $\text{meas}(\Omega \backslash \Omega_\text{s}) \longrightarrow 0$ as $s \longrightarrow \infty$.
By (A_5) there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular converge in $W_0^1 L_M(\Omega)$.
Here, we define

$$w_{n,j}^h = T_{2k}(u_n - v_0 - T_h(u_n - v_0) + T_k(u_n) - T_k(v_j)),$$

$$w_j^h = T_{2k}(u - v_0 - T_h(u - v_0) + T_k(u) - T_k(v_j))$$

and

$$w^h = T_{2k}(u - v_0 - T_h(u - v_0))$$

where $h > 2k > 0$.
For $\eta = \exp(-4\gamma \text{k}^2)$, we defined the following function as

$$v_{n,j}^h = u_n - \eta \varphi_k(w_{n,j}^h). \tag{2.3.27}$$

We take $v_{n,j}^h$ as test function in (P_n), we obtain,

$$\langle A(u_n), \eta \varphi_k(w_{n,j}^h) \rangle + \int_\Omega g_n(x, u_n, \nabla u_n) \eta \varphi_k(w_{n,j}^h)\, dx \leq \int_\Omega f_n \eta \varphi_k(w_{n,j}^h)\, dx.$$

Which, implies that

$$\langle A(u_n), \varphi_k(w_{n,j}^h) \rangle + \int_\Omega g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h)\, dx \leq \int_\Omega f_n \varphi_k(w_{n,j}^h)\, dx. \tag{2.3.28}$$

It follows that

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \nabla w_{n,j}^h \varphi_k'(w_{n,j}^h)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h)\, dx \\ \leq \int_\Omega f_n \varphi_k(w_{n,j}^h)\, dx. \end{aligned} \tag{2.3.29}$$

Note that, $\nabla w_{n,j}^h = 0$ on the set where $|u_n| > h + 5k$, therefore, setting $m = 5k + h$. We get, by (2.3.29),

$$\begin{aligned} \int_\Omega a(x, T_m(u_n), \nabla T_m(u_n)) \nabla w_{n,j}^h \varphi_k'(w_{n,j}^h)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) \varphi_k(w_{n,j}^h)\, dx \\ \leq \int_\Omega f_n \varphi_k(w_{n,j}^h)\, dx. \end{aligned}$$

In view of (2.3.20), we have $\varphi_k(w_{n,j}^h) \to \varphi_k(w_j^h)$ weakly* as $n \to +\infty$ in $L^\infty(\Omega)$, and then

$$\int_\Omega f_n \varphi_k(w_{n,j}^h)\, dx \to \int_\Omega f \varphi_k(w_j^h)\, dx \quad \text{as} \quad n \to +\infty.$$

Again tends j to infinity, we get

$$\int_\Omega f \varphi_k(w_j^h)\, dx \to \int_\Omega f \varphi_k(w^h)\, dx \quad \text{as} \quad j \to +\infty,$$

finally letting h to infinity, we deduce by using the Lebesgue Theorem $\int_\Omega f\varphi_k(w^h)\,dx \to 0$.
So that

$$\int_\Omega f_n\varphi_k(w^h_{n,j})\,dx = \epsilon(n,j,h).$$

Since in the set $\{x\in\Omega, |u_n(x)|>k\}$, we have $g_n(x,u_n,\nabla u_n)\varphi_k(w^h_{n,j})\geq 0$, we deduce from (2.3.29) that

$$\begin{aligned}&\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi_k'(w^h_{n,j})\,dx\\ &\qquad+\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w^h_{n,j})\,dx\leq\epsilon(n,j,h).\end{aligned} \tag{2.3.30}$$

Splitting the first integral on the left hand side of (2.3.30) where $|u_n|\leq k$ and $|u_n|>k$, we can write,

$$\begin{aligned}&\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi_k'(w^h_{n,j})\,dx\\ &\quad=\int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi_k'(w^h_{n,j})\,dx\\ &\quad+\int_{\{|u_n|>k\}} a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi_k'(w^h_{n,j})\,dx.\end{aligned} \tag{2.3.31}$$

The first term of the right hand side of the last inequality can write as

$$\begin{aligned}&\int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi_k'(w^h_{n,j})\,dx\\ &\quad\geq\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi_k'(w^h_{n,j})\,dx\\ &\quad-\varphi_k'(2k)\int_{\{|u_n|>k\}}|a(x,T_k(u_n),0)||\nabla v_j|\,dx.\end{aligned} \tag{2.3.32}$$

Recalling that, $|a(x,T_k(u_n),0)|\chi_{\{|u_n|>k\}}$ converges to $|a(x,T_k(u),0)|\chi_{\{|u|>k\}}$ strongly in $L_M(\Omega)$, moreover, since $|\nabla v_j|$ modular converges to $|\nabla T_k(u)|$, then

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}}|a(x,T_k(u_n),0)||\nabla v_j|\,dx=\epsilon(n,j).$$

For the second term of the right hand side of (2.3.31) we can write, using (A_3)

$$\begin{aligned}&\int_{\{|u_n|>k\}} a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^h_{n,j}\varphi_k'(w^h_{n,j})\,dx\\ &\quad\geq-\varphi_k'(2k)\int_{\{|u_n|>k\}}|a(x,T_m(u_n),\nabla T_m(u_n))||\nabla v_j|\,dx\\ &\quad-\varphi_k'(2k)\int_{\{|u_n-v_0|>h\}}\delta(x)\,dx.\end{aligned} \tag{2.3.33}$$

Since $|a(x,T_m(u_n),\nabla T_m(u_n))|$ is bounded in $L_{\overline{M}}(\Omega)$, we have, for a subsequence

$$|a(x,T_m(u_n),\nabla T_m(u_n))|\rightharpoonup l_m$$

weakly in $L_{\overline{M}}(\Omega)$ for $\sigma(L_{\overline{M}}, E_M)$ as n tends to infinity , and since

$$\nabla v_j \chi_{\{|u_n|>k\}} \to \nabla v_j \chi_{\{|u|>k\}}$$

strongly in $E_M(\Omega)$ as n tends to infinity, we have

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))||\nabla v_j| \, dx \to -\varphi_k'(2k)\int_{\{|u|>k\}} l_m|\nabla v_j| \, dx$$

as n tends to infinity.
Using now, the modular convergence of (v_j), we get

$$-\varphi_k'(2k)\int_{\{|u|>k\}} l_m|\nabla v_j| \, dx \to -\varphi_k'(2k)\int_{\{|u|>k\}} l_m|\nabla T_k(u)| \, dx = 0$$

as j tends to infinity.
Finally

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))||\nabla v_j| \, dx = \epsilon_h(n, j). \tag{2.3.34}$$

On the other hand, since $\delta \in L^1(\Omega)$ it is easy to see that

$$-\varphi_k'(2k)\int_{\{|u_n-v_0|>h\}} \delta(x) \, dx = \epsilon(n, h). \tag{2.3.35}$$

Combining (2.3.32), ..., (2.3.35), we deduce

$$\begin{aligned}
&\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_{n,j}^h \varphi_k'(w_{n,j}^h) \, dx \\
&\qquad \geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(v_j)]\varphi_k'(w_{n,j}^h) \, dx \\
&\qquad +\epsilon(n, h) + \epsilon(n, j) + \epsilon_h(n, j).
\end{aligned} \tag{2.3.36}$$

Which implies that

$$\begin{aligned}
&\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_{n,j}^h \varphi_k'(w_{n,j}^h) \, dx \\
&\qquad \geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\
&\qquad \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{n,j}^h) \, dx \\
&\qquad +\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{n,j}^h) \, dx \\
&\qquad -\int_{\Omega\setminus\Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\varphi_k'(w_{n,j}^h) \, dx \\
&\qquad +\epsilon(n, h) + \epsilon(n, j) + \epsilon_h(n, j)
\end{aligned} \tag{2.3.37}$$

where χ_s^j denotes the characteristic function of the subset

$$\Omega_s^j = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}.$$

By (2.3.26) and the fact that $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_{n,j}^h)$ tends to $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_j^h)$ strongly in $(E_M(\Omega))^N$, the third term of the right hand side of (2.3.37) tends to quantity

$$\int_\Omega h_k \nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_j^h)\, dx$$

as n tends to infinity.
Letting now j tends to infinity, by using the modular convergence of v_j, we have

$$\int_\Omega h_k \nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_j^h)\, dx \to \int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)\varphi'_k(w^h)\, dx$$

as j tends to infinity.
Finally

$$\begin{aligned} -\int_{\Omega\setminus\Omega_s^j} & a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\varphi'_k(w_{n,j}^h)\, dx \\ &= -\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(v_j)\varphi'_k(w^h)\, dx + \epsilon_h(n,j). \end{aligned} \tag{2.3.38}$$

Concerning the second term of the right hand side of the (2.3.37) we can write

$$\begin{aligned} \int_\Omega & a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\varphi'_k(w_{n,j}^h)\, dx \\ &= \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\varphi'_k(T_k(u_n) - T_k(v_j))\, dx \\ &\quad -\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\varphi'_k(w_{n,j}^h)\, dx. \end{aligned} \tag{2.3.39}$$

The first term of the right hand side of (2.3.39) tends to the quantity

$$\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u)\varphi'_k(T_k(u) - T_k(v_j))\, dx \quad \text{as} \quad n\to\infty$$

since

$$\begin{aligned} & a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\varphi'_k(T_k(u_n) - T_k(v_j)) \\ &\qquad \to a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\varphi'_k(T_k(u) - T_k(v_j)) \end{aligned}$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by Lemma 1.1.6 of Chapter I and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ for $\sigma(\prod L_M, \prod E_{\overline{M}})$.
For the second term of the right hand side of the (2.3.39) it is easy to see that

$$\begin{aligned} \int_\Omega & a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\varphi'_k(w_{n,j}^h)\, dx \\ &\to \int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\varphi'_k(w_j^h)\, dx. \end{aligned} \tag{2.3.40}$$

as $n\to\infty$. Consequently, we have

$$\begin{aligned} \int_\Omega & a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\varphi'_k(w_{n,j}^h)\, dx \\ &= \int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j]\varphi'_k(w_j^h)\, dx + \epsilon_{j,h}(n) \end{aligned} \tag{2.3.41}$$

since

$$\nabla T_k(v_j)\chi_s^j\varphi'(w_j^h) \to \nabla T_k(u)\chi_s\varphi_k'(w^h)$$

strongly in $(E_M(\Omega))^N$ as $j \to +\infty$, it is easy to see that

$$\begin{aligned}\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_j^h)\, dx \\ \to \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\varphi_k'(0)\, dx\end{aligned}$$

as $j \to +\infty$ thus

$$\begin{aligned}\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{n,j}^h)\, dx \\ = \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\varphi_k'(0)\, dx + \epsilon(n, j)\end{aligned} \tag{2.3.42}$$

Combining (2.3.37), (2.3.38) and (2.3.42), we get

$$\begin{aligned}&\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w_{n,j}^h\varphi_k'(w_{n,j}^h)\, dx \\ &\qquad \geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\ &\qquad \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{n,j}^h)\, dx \\ &+\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi_k'(0)\, dx + \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\varphi_k'(0)\, dx + \epsilon(n, j, h).\end{aligned} \tag{2.3.43}$$

We now, turn to the second term of the left hand side of (2.3.30), we have

$$\begin{aligned}&\left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,j}^h)\, dx\right| \\ &\qquad \leq b(k)\int_\Omega (c(x) + M(\nabla T_k(u_n)))|\varphi_k(w_{n,j}^h)|\, dx \\ &\qquad \leq b(k)\int_\Omega c(x)|\varphi_k(w_{n,j}^h)|\, dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,j}^h)| \\ &\qquad +\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,j}^h)|\, dx \\ &\qquad -\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla v_0|\varphi_k(w_{n,j}^h)|\, dx \\ &\qquad \leq \epsilon(n, j, h) + \tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,j}^h)|\, dx.\end{aligned} \tag{2.3.44}$$

The last term of the last side of this inequality reads as

$$\begin{aligned}&\tfrac{b(k)}{\alpha}\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)] \\ &\qquad \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{n,j}^h)|\, dx \\ &\qquad +\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{n,j}^h)|\, dx \\ &\qquad -\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\chi_s^j|\varphi_k(w_{n,j}^h)|\, dx\end{aligned} \tag{2.3.45}$$

and reasoning as above, it is easy to see that

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{n,j}^h)|\,dx=\epsilon(n,j)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_s^j|\varphi_k(w_{n,j}^h)|\,dx=\epsilon(n,j,h).$$

So that

$$\begin{aligned}\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,j}^h)\,dx\right|&\\ \le \tfrac{b(k)}{\alpha}\int_\Omega &[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{n,j}^h)|\,dx+\epsilon(n,j,h).\end{aligned}\tag{2.3.46}$$

Combining (2.3.30), (2.3.43) and (2.3.46), we obtain

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j](\varphi_k'(w_{n,j}^h)-\tfrac{b(k)}{\alpha}|\varphi_k(w_{n,j}^h)|)\,dx\\ &\le\int_{\Omega\setminus\Omega_s}h_k\nabla T_k(u)\varphi_k'(0)\,dx+\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)\,dx+\epsilon(n,j,h).\end{aligned}\tag{2.3.47}$$

Which implies that, by using (2.3.11)

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &\le 2\int_{\Omega\setminus\Omega_s}h_k\nabla T_k(u)\varphi_k'(0)\,dx+2\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)\,dx+\epsilon(n,j,h).\end{aligned}\tag{2.3.48}$$

Now, remark that

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\ &=\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &\qquad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\ &\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s]\,dx.\end{aligned}\tag{2.3.49}$$

We shall pass to the limit in n and j in the last three terms of the right hand side of the last inequality, we get

$$\begin{aligned}\int_\Omega &a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &=\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\nabla T_k(u)\,dx+\epsilon(n,j),\end{aligned}$$

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx$$
$$=\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\,dx+\epsilon(n),$$

and
$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s]\,dx=\epsilon(n,j),$$
which implies that

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\ &=\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx+\epsilon(n,j).\end{aligned}\tag{2.3.50}$$

Combining (2.3.48) and (2.3.50), we have

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\ &\le 2\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\varphi_k'(0)\,dx+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)\,dx+\epsilon(n,j,h).\end{aligned}\tag{2.3.51}$$

By passing to the lim sup over n, and letting j,h,s tend to infinity, we obtain

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx=0,$$

thus implies by using the Lemma 2.3.1 that

$$M(|\nabla T_k(u_n)|)\to M(|\nabla T_k(u)|)\ \text{ in }\ L^1(\Omega).\tag{2.3.52}$$

and then
$$\nabla u_n\to\nabla u\ \text{ a.e. in }\ \Omega.\tag{2.3.53}$$

Step 6. Equi-integrability of the nonlinearities.

We need to prove that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\ \text{ strongly in }\ L^1(\Omega),\tag{2.3.54}$$

in particular it is enough to prove the equiintegrable of $g_n(x,u_n,\nabla u_n)$. To this purpose. We take $u_n-T_1(u_n-v_0-T_h(u_n-v_0))$ as test function in (P_n), we obtain

$$\int_{\{|u_n-v_0|>h+1\}}|g_n(x,u_n,\nabla u_n)|\,dx\le\int_{\{|u_n-v_0|>h\}}(|f_n|+\delta(x))\,dx.$$

Let $\varepsilon>0$, then there exists $h(\varepsilon)\ge 1$ such that

$$\int_{\{|u_n-v_0|>h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|\,dx<\varepsilon/2.\tag{2.3.55}$$

For any measurable subset $E \subset \Omega$, we have

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \int_E b(h(\varepsilon)+\|v_0\|_\infty)\Big(c(x)+M(\nabla T_{h(\varepsilon)+\|v_0\|_\infty}(u_n))\Big)\,dx + \int_{\{|u_n-v_0|>h(\varepsilon)\}} |g(x,u_n,\nabla u_n)|\,dx.$$

In view of (2.3.52) there exists $\eta(\varepsilon)>0$ such that

$$\int_E b(h(\varepsilon)+\|v_0\|_\infty)\Big(c(x)+M(\nabla T_{h(\varepsilon)+\|v_0\|_\infty}(u_n))\Big)\,dx < \varepsilon/2 \quad \text{for all E such that } |E|<\eta(\varepsilon). \tag{2.3.56}$$

Finally, combining (2.3.55) and (2.3.56), one easily has

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx < \varepsilon \ \text{ for all } E \text{ such that } |E|<\eta(\varepsilon),$$

which implies (2.3.54).

Step 7. Passing to the limit.

Let $v \in K_\psi \cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$, we take $u_n - T_k(u_n-v)$ as test function in (P_n), we can write

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx \le \int_\Omega f_nT_k(u_n-v)\,dx. \tag{2.3.57}$$

which implies that

$$\int_{\{|u_n-v|\le k\}} a(x,u_n,\nabla u_n)\nabla(u_n-v_0)\,dx + \int_{\{|u_n-v|\le k\}} a(x,T_{k+\|v\|_\infty}u_n,\nabla T_{k+\|v\|_\infty}(u_n))\nabla(v_0-v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx \le \int_\Omega f_nT_k(u_n-v)\,dx. \tag{2.3.58}$$

By Fatou's Lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$ on easily see that

$$\int_{\{|u-v|\le k\}} a(x,u,\nabla u)\nabla(u-v_0)\,dx + \int_{\{|u-v|\le k\}} a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla(v_0-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \le \int_\Omega fT_k(u-v)\,dx. \tag{2.3.59}$$

Hence

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx. \tag{2.3.60}$$

Now, let $v \in K_\psi \cap L^\infty(\Omega)$, by the condition (A_5) there exists $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v modular, let $h > \max(\|v_0\|_\infty, \|v\|_\infty)$, taking $v = T_h(v_j)$ in (2.3.60), we have

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(v_j))\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-T_h(v_j))\,dx \leq \int_\Omega f T_k(u-T_h(v_j))\,dx. \tag{2.3.61}$$

We can easily pass to the limit as $j \to +\infty$ to get

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(v))\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-T_h(v))\,dx \leq \int_\Omega f T_k(u-T_h(v))\,dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \tag{2.3.62}$$

the same, since $h \geq \max(\|v_0\|_\infty, \|v\|_\infty)$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \quad \forall\, k > 0. \tag{2.3.63}$$

thus, the proof of the Theorem 2.2.1 is now complete.

Remark 2.3.2. *the result of our Theorem remain valid if we replace (A_4) by the following conditions*

(A_6) *There exists a strictly positive constant α such that for a.e. $x \in \Omega$ and for all $(s,\xi) \in I\!R \times I\!R^N$*

$$a(x,s,\xi)\xi \geq \alpha M(\xi),$$

(A_7) $K_\psi \cap L^\infty(\Omega) \neq \emptyset$.

Chapter 3

STRONGLY NONLINEAR ELLIPTIC UNILATERAL PROBLEMS WITH NATURAL GROWTH IN ORLICZ SPACES [1]

In this Chapter, we shall be concerned with the existence result of unilateral problem associated to the equations of the form,

$$Au + g(x, u, \nabla u) = \mu,$$

where A is a Leray-Lions operator from its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1}E_{\overline{M}}(\Omega)$. On the nonlinear lower order term $g(x, u, \nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, and satisfies the sign condition. The right hand side μ belong either to $L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$ or to $W^{-1}E_{\overline{M}}(\Omega)$.

3.1. Introduction

Let Ω be an open bounded subset of $I\!\!R^N, N \geq 2$, with segment property. Let us consider the following nonlinear Dirichlet problem

$$-\text{div}(a(x, u, \nabla u)) + g(x, u, \nabla u) = \mu \tag{3.1.1}$$

$Au = -\text{div}a(x, u, \nabla u)$ is a Leray-Lions operator defined on its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$, with M an N-function and where g is a nonlinearity with the "natural" growth condition:

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|))$$

and which satisfies the classical sign condition

$$g(x, s, \xi).s \geq 0.$$

[1]Abs. and App. Anal. Vol. 2006, Art. ID 46867, pp 1-20 DOI 10.1155/AAA/2006/46867.

Our first aim of this Chapter is to prove (in Theorem 3.2.1) the existence result for the following unilateral problem

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega),\ u \geq \psi,\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\ dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\ dx \leq \langle \mu, T_k(u-v)\rangle, \\ v \in K_\psi \cap L^\infty(\Omega)\ \forall k > 0, \end{cases}$$

where $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$ and where $K_\psi = \{u \in W_0^1L_M(\Omega),\ u \geq \psi\ \ a.e. \text{ in } \Omega\}$. The second aim of this Chapter, is to prove (in Theorem 3.3.1) an existence result for unilateral problems associated to (3.1.1) in the setting of the Orlicz-Sobolev space, where μ belongs to $W^{-1}E_{\overline{M}}(\Omega)$, More precisely, we deals with the existence of solutions to the following problem

$$\begin{cases} u \in K_\psi,\ g(x,u,\nabla u) \in L^1(\Omega),\ g(x,u,\nabla u)u \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla(u-v)\ dx + \int_\Omega g(x,u,\nabla u)(u-v)\ dx \leq \langle \mu, u-v\rangle, \\ v \in K_\psi \cap L^\infty(\Omega). \end{cases}$$

3.2. Non variational problem

3.2.1. Statement of result

In this Subsection we assume that $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$. Consider the nonlinear problem with Dirichlet boundary conditions,

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega),\ u \geq \psi,\ g(x,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\ dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\ dx \\ \qquad\qquad \leq \langle \mu, T_k(u-v)\rangle, \\ \forall\ \ v \in K_\psi \cap L^\infty(\Omega),\ \forall\ k > 0. \end{cases} \tag{3.2.1}$$

We have the following result:

Theorem 3.2.1. *Under the assumptions (A_1) - (A_5), (G_1) and (G_2) there exists at least one solution of (3.2.1)*

3.2.2. Approximate problems and apriorie estimate

Since $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$ then μ can be write as follows

$$\mu = f - \text{div}F \text{ with } \ f \in L^1(\Omega),\ F \in (E_{\overline{M}}(\Omega))^N.$$

Let us define

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}.$$

and let us consider the approximate unilateral problems:

$$\begin{cases} u_n \in K_\psi \cap \mathcal{D}(A), \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \\ \qquad\qquad \displaystyle\leq \int_\Omega f_n(u_n - v)\,dx + \int_\Omega F\nabla(u_n - v)\,dx, \\ \forall v \in K_\psi. \end{cases} \tag{3.2.2}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$.
Applying Lemma 2.2.1, the problem (3.2.2) has at least one solution.

Proposition 3.2.1. *Assume that the assumptions $(A_1)-(A_5)$, G_1 and G_2 hold. Then there exists a function $u \in \mathcal{T}_0^{1,M}(\Omega)$ and $h_k \in (L_{\overline{M}}(\Omega))^N$ such that for all $k > 0$ we have the following property:*

1) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^1L_M(\Omega)$ *for* $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$
$T_k(u_n) \to T_k(u)$ *strongly in* $E_M(\Omega)$ *and a.e. in* Ω.

2) $a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k$ *weakly in* $(L_{\overline{M}}(\Omega))^N$ *for* $\sigma(\Pi L_{\overline{M}}(\Omega), \Pi E_M(\Omega))$.

3) $\displaystyle\int_\Omega |g_n(x,u_n,\nabla u_n)|\,dx \leq C.$

Proof. The proof of this Proposition is inspired of the proof of Theorem 2.2.1 of Chapter II.

3.2.3. Modular convergence of the truncation

Proposition 3.2.2. *Let u_n be a solutions of the problem (3.2.2), then for all $k > 0$, we have*

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \quad \text{in} \quad L^1(\Omega).$$

Proof of Proposition 3.2.2

STEP 1. We prove that $\nabla u_n \to \nabla u$ a.e. in Ω.

Let $k > \|v_0\|_\infty$. By the condition (A_5) there exists a sequence $v_j \in K_\psi \cap W_0^1E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1L_M(\Omega)$.
Consider the expression

$$I_{n,r} = \int_{\Omega_r} \{[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\}^\theta\,dx$$

where $0 < \theta < 1$. Let A_n be expression in brace above, then for any $0 < \eta < 1$

$$I_{n,r} = \int_{\Omega_r \cap \{|T_k(u_n) - T_k(v_j)| \leq \eta\}} A_n^\theta\,dx + \int_{\Omega_r \cap \{|T_k(u_n) - T_k(v_j)| > \eta\}} A_n^\theta\,dx.$$

Since $a(x, T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$ (see Proposition 3.4.1), while $\nabla T_k(u_n)$ bounded in $(L_M(\Omega))^N$, then by applying the Hölder's inequality, we obtain

$$I_{n,r} \leq c_1 \left(\int_{\Omega_r \cap \{|T_k(u_n) - T_k(v_j)| \leq \eta\}} A_n \, dx \right)^{\theta} + c_2 \text{meas}\{x \in \Omega : |T_k(u_n) - T_k(v_j)| > \eta\}^{1-\theta}. \tag{3.2.3}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\Omega_r \cap \{|T_k(u_n) - T_k(v_j)| \leq \eta\}} A_n \, dx \\
&\quad \leq \int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \, dx \\
&\quad = \int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \, dx \\
&\quad - \int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \\
&\quad + \int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \, dx \\
&\quad - \int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u)\chi_s \, dx.
\end{aligned} \tag{3.2.4}$$

Since $a(x, T_k(u_n), \nabla T_k(u_n))$ bounded in $(L_{\overline{M}}(\Omega))^N$, then

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \rho_k \quad \text{weakly in} \quad (L_{\overline{M}}(\Omega))^N \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}})$$

as $n \to +\infty$ for some $\rho_k \in (L_{\overline{M}}(\Omega))^N$. We deduce that

$$\begin{aligned}
&\int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u)\chi_s \, dx \\
&\qquad \longrightarrow \int_{\{|T_k(u) - T_k(v_j)| \leq \eta\}} \rho_k \nabla T_k(u)\chi_s \, dx,
\end{aligned} \tag{3.2.5}$$

$$\begin{aligned}
&\int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \, dx \\
&\qquad \longrightarrow \int_{\{|T_k(u) - T_k(v_j)| \leq \eta\}} \rho_k \nabla T_k(v_j) \, dx,
\end{aligned} \tag{3.2.6}$$

as $n \to \infty$, letting also j to infinity, we get

$$\begin{aligned}
&\int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u)\chi_s \, dx \\
&\qquad = \int_{\Omega} \rho_k \nabla T_k(u)\chi_s \, dx + \varepsilon(n, j),
\end{aligned} \tag{3.2.7}$$

and

$$\begin{aligned}
&\int_{\{|T_k(u_n) - T_k(v_j)| \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \, dx \\
&\qquad = \int_{\Omega} \rho_k \nabla T_k(u) \, dx + \varepsilon(n, j).
\end{aligned} \tag{3.2.8}$$

Concerning the second term of the right hand side of the (3.2.4), it is easy to see that

$$\begin{aligned}\int_{\{|T_k(u_n)-T_k(v_j)|\leq\eta\}} & a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s])\,dx \\ &= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\,dx+\varepsilon(n,j).\end{aligned} \tag{3.2.9}$$

On the other side, the use a test function $u_n - T_\eta(u_n - T_k(v_j))$ in (3.2.2), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_\eta(u_n-T_k(v_j))\,dx \\ \leq \int_\Omega f_n T_\eta(u_n-T_k(v_j))\,dx + \int_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx,\end{aligned}$$

which implies, by using the assertion 3) of Proposition 3.3.1

$$\begin{aligned}\int_\Omega & a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx \\ &\leq c_1\eta + \int_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx.\end{aligned} \tag{3.2.10}$$

Splitting the integral on the left hand side of the last inequality where $|u_n| \leq k$ and $|u_n| > k$, we can write,

$$\begin{aligned}\int_\Omega & a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx \\ &= \int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx \\ &+ \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx,\end{aligned}$$

which implies, by using (A_4)

$$\begin{aligned}\int_\Omega & a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx \\ &\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx \\ &- \int_{\{|u_n|>k\}} (|a(x,T_k(u_n),0)|+|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|)|\nabla v_j|\,dx \\ &- \int_{\{|u_n-T_k(v_j)|\leq\eta,|u_n|>k\}} |a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))||\nabla v_0|\,dx \\ &- \int_{\{|u_n-T_k(v_j)|\leq\eta,|u_n|>k\}} \delta(x)\,dx.\end{aligned} \tag{3.2.11}$$

Combining (3.2.10) and (3.2.11), we deduce

$$\begin{aligned}\int_\Omega & a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx \\ &\leq \int_{\{|u_n|>k\}} (|a(x,T_k(u_n),0)|+|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|)|\nabla v_j|\,dx \\ &+ \int_{\{|u_n-T_k(v_j)|\leq\eta,|u_n|>k\}} |a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))||\nabla v_0|\,dx \\ &+ \int_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx \\ &+ \int_{\{|u_n|>k,|u_n-T_k(v_j)|\leq\eta\}} \delta(x)\,dx + c_1\eta.\end{aligned} \tag{3.2.12}$$

Since $|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|$ bounded in $L_{\overline{M}}(\Omega)$ there exists a function $h_k \in L_{\overline{M}}(\Omega)$ such that
$|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|+|a(x,T_k(u_n),0)| \rightharpoonup h_k+|a(x,T_k(u),0)|$ for $\sigma(Ł_{\overline{M}}(\Omega),E_M(\Omega))$ as $n\to+\infty$, while $|\nabla v_j|\chi_{\{|u_n|>k\}} \to |\nabla v_j|\chi_{\{|u|>k\}}$ strongly in $E_M(\Omega)$, and by the modular convergence of v_j, we deduce that the first term of the right hand side of the (3.2.12) tends to 0 as $n\to\infty$ and $j\to\infty$.
Similarly, we can easy to pass to the limit on other terms in (3.2.12) as $n\to\infty$ and $j\to\infty$, we get.

$$\begin{aligned}\int_{\{|u_n-T_k(v_j)|\le\eta,|u_n|>k\}} &|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))||\nabla v_0|\,dx\\ &= \int_{\{|u-T_k(u)|\le\eta,|u|>k\}} h_k|\nabla v_0|\,dx+\varepsilon(n,j),\end{aligned} \tag{3.2.13}$$

$$\begin{aligned}\int_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx &= \int_{\{|u-T_k(u)|\le\eta,|u|>k\}} F\nabla T_\eta(u-T_k(u))\,dx+\varepsilon(n,j)\\ &\le c_3\|F\chi_{\{|u-T_k(u)|\le\eta,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M+\varepsilon(n,j),\end{aligned} \tag{3.2.14}$$

and

$$\int_{\{|u_n|>k,|u_n-T_k(v_j)|\le\eta\}} \delta(x)\,dx = \int_{\{|u|>k,|u-T_k(u)|\le\eta\}} \delta(x)\,dx+\varepsilon(n,j). \tag{3.2.15}$$

Consequently, we deduce

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))&\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx\\ &\le \int_{\{|u-T_k(u)|\le\eta,|u|>k\}} h_k|\nabla v_0|\,dx+\int_{\{|u|>k,|u-T_k(u)|\le\eta\}}\delta(x)\,dx\\ &+c_3\|F\chi_{\{|u-T_k(u)|\le\eta,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M+\varepsilon(n,j),\end{aligned} \tag{3.2.16}$$

hence, from (3.2.4), (3.2.7), (3.2.8), (3.2.9) and (3.2.16), we get

$$\begin{aligned}\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} &A_n\,dx\\ &\le \int_{\Omega\setminus\Omega_s}\rho_k\nabla T_k(u)\,dx+\int_{\Omega\setminus\Omega_s} a(x,T_k(u_n),0)\nabla T_k(u)\,dx\\ &+\int_{\{|u-T_k(u)|\le\eta,|u|>k\}} h_k|\nabla v_0|\,dx+\int_{\{|u|>k,|u-T_k(u)|\le\eta\}}\delta(x)\,dx\\ &+c_3\|F\chi_{\{|u-T_k(u)|\le\eta,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M+C_1\eta+\varepsilon(n,j).\end{aligned} \tag{3.2.17}$$

Finally, in virtu of (3.2.3) and (3.2.17), we deduce

$$\begin{aligned}I_{n,r}\le \Big\{C_2&\int_{\Omega\setminus\Omega_s}\rho_k\nabla T_k(u)\,dx+\int_{\Omega\setminus\Omega_s} a(x,T_k(u_n),0)\nabla T_k(u)\,dx\\ &+\int_{\{|u-T_k(u)|\le\eta,|u|>k\}} h_k|\nabla v_0|\,dx+\int_{\{|u|>k,|u-T_k(u)|\le\eta\}}\delta(x)\,dx\\ &+c_3\|F\chi_{\{|u-T_k(u)|\le\eta,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M+C_1\eta+\varepsilon(n,j)\Big\}^\theta\\ &+C_4\text{meas}\{x:|T_k(u_n)-T_k(v_j)|>\eta\}^{1-\theta},\end{aligned}$$

which implies, by passing to the limit sup over n

$$\begin{aligned}\limsup_{n\to\infty} I_{n,r} \le \Big\{ & C_2\int_{\Omega\setminus\Omega_s)} \rho_k \nabla T_k(u)\,dx + \int_{\Omega\setminus\Omega_s} a(x,T_k(u_n),0)\nabla T_k(u)\,dx \\ & + \int_{\{|u-T_k(u)|\le\eta,|u|>k\}} h_k|\nabla v_0|\,dx + \int_{\{|u|>k,|u-T_k(u)|\le\eta\}} \delta(x)\,dx \\ & + c_3\|F\chi_{\{|u-T_k(u)|\le\eta,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M + C_1\eta + \varepsilon(n,j)\Big\}^\theta \\ & + c_4 \text{meas}\{x : |T_k(u)-T_k(v_j)|>\eta\}^{1-\theta},\end{aligned}$$

and also, letting $j\to\infty$ and $s\to\infty$, η tend to 0, we get

$$\limsup_{n\to\infty} I_{n,r} = 0.$$

Since r and k are arbitrary, we have for a subsequence

$$\nabla u_n \to \nabla u \ \ a.e. \ \text{ in } \ \Omega. \tag{3.2.18}$$

STEP 2. We claim that $M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|)$ **in** $L^1(\Omega)$.

We fix $k > \|v_0\|_\infty$. By the condition (A_5) there exists a sequence $v_j \in K_\psi \cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1L_M(\Omega)$.
Here, we define

$$w_{n,j}^h = T_{2k}(u_n - v_0 - T_h(u_n-v_0) + T_k(u_n) - T_k(v_j)),$$

$$w_j^h = T_{2k}(u - v_0 - T_h(u-v_0) + T_k(u) - T_k(v_j))$$

and

$$w^h = T_{2k}(u - v_0 - T_h(u-v_0))$$

where $h > 2k > 0$.
For $\eta = \exp(-4\gamma \mathrm{k}^2)$, we defined the following function as

$$v_{n,j}^h = u_n - \eta\varphi_k(w_{n,j}^h). \tag{3.2.19}$$

We take $v_{n,j}^h$ as test function in (3.2.2), and reasoning as in Chapter II, we have

$$\begin{aligned}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx \\ \le 2\int_{\Omega\setminus\Omega_s} \rho_k\nabla T_k(u)\varphi_k'(0)\,dx \\ +2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\varphi_k'(0)\,dx \\ +2\int_\Omega F\nabla T_{2k}(u-v_0-T_h(u-v_0))\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx + \epsilon(n,j,h).\end{aligned} \tag{3.2.20}$$

Now, we chow that

$$\lim_{h\to\infty}\int_\Omega F\nabla T_{2k}(u-v_0-T_h(u-v_0))\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx = 0. \tag{3.2.21}$$

Indeed, we have

$$
\begin{aligned}
\Big|\int_\Omega F\nabla T_{2k}(u-v_0-T_h(u-v_0))\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx\Big| \\
\leq \int_{\{h<|u-v_0|\leq h+2k\}} |F||\nabla u|\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx \\
+\int_{\{|u-v_0|>h\}} |F||\nabla v_0|\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx,
\end{aligned}
\tag{3.2.22}
$$

which implies, by using the Young's inequality,

$$
\begin{aligned}
\Big|\int_\Omega F\nabla T_{2k}(u-v_0-T_h(u-v_0))\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx\Big| \\
\leq \int_{\{|u-v_0|>h\}} |F||\nabla v_0|\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx \\
+C(k)\int_{\{|u-v_0|>h\}} \overline{M}(|F|)\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx \\
+\alpha\int_{\{h<|u-v_0|\leq h+2k\}} M(|\nabla u|)\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx.
\end{aligned}
\tag{3.2.23}
$$

It remains to show, for our purposes, that the all term on the right hand side of (3.2.23) converge to zero as h goes to infinity. The only difficulty that exists is in the last term. For the other terms it suffices to apply Lebesgue's Theorem.
We deal now with this term. Let us observe that, for η small enough if we take $u_n-\eta\varphi_k(T_{2k}(u_n-v_0-T_h(u_n-v_0)))$ as test function in (3.2.2), we obtain

$$
\begin{aligned}
\int_{\{h\leq|u_n-v_0|\leq 2k+h\}} a(x,u_n\nabla u_n)\nabla(u_n-v_0)\varphi_k'(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx \\
+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx \\
\leq \int_{\{h\leq|u_n-v_0|\leq 2k+h\}} F\nabla(u_n-v_0)\varphi_k'(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx \\
\int_\Omega f_n\varphi_k(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx.
\end{aligned}
$$

Since $g_n(x,u_n,\nabla u_n)\varphi_k(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\geq 0$, We get

$$
\begin{aligned}
\int_{\{h\leq|u_n-v_0|\leq 2k+h\}} a(x,u_n\nabla u_n)\nabla(u_n-v_0)\varphi_k'(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx \\
\leq \int_{\{h\leq|u_n-v_0|\leq 2k+h\}} F\nabla(u_n-v_0)\varphi_k'(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx \\
\int_\Omega f_n\varphi_k(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx.
\end{aligned}
$$

Which yields, thanks to (A_4) and Young's inequalities

$$
\begin{aligned}
\tfrac{\alpha}{2}\int_{\{h\leq|u_n-v_0|\leq 2k+h\}} M(|\nabla u_n|)\varphi_k'(T_{2k}(u_n-v_0-T_h(u_n-v_0)))\,dx \\
\leq C_1(k)\int_{\{|u_n-v_0|>h\}} |F||\nabla v_0|\,dx + C_2(k)\int_{\{|u_n-v_0|>h\}} \overline{M}(|F|)\,dx \\
+C_3(k)\int_{\{|u_n-v_0|>h\}} (|f_n|+\delta(x))\,dx.
\end{aligned}
$$

Letting n to infinity, by using the Fatou's Lemma, we get

$$\begin{aligned}\tfrac{\alpha}{2}\int_{\{h\le|u-v_0|\le 2k+h\}} & M(|\nabla u|)\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx\\ &\le C_1(k)\int_{\{|u-v_0|>h\}}|F||\nabla v_0|\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx\\ &\quad + C_2(k)\int_{\{|u-v_0|>h\}}(|f|+\delta(x))\,dx.\end{aligned}$$

Consequently, we have, as h tend to infinity,

$$\limsup_{h\to\infty}\int_{\{h\le|u-v_0|\le 2k+h\}} M(|\nabla u|)\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx = 0,$$

so that

$$\lim_{h\to\infty}\int_\Omega F\nabla T_{2k}(u-v_0-T_h(u-v_0))\varphi_k'(T_{2k}(u-v_0-T_h(u-v_0)))\,dx = 0$$

which implies (3.2.21).
Finally, if we pass to the limit in (3.2.20) as h and s to infinity, we get

$$\begin{aligned}\limsup_{s\to\infty}\limsup_{n\to\infty}\int_\Omega & [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx = 0\end{aligned}$$

and by using the Lemma 2.3.1, we can conclude the result.

Proof of Theorem 3.2.1

Thanks to Proposition 3.2.1 and 3.2.2 we can pass to the limit in (3.2.2) by using the same technique in steps 7 and 8 of the proof of Theorem 3.3.1 of Chapter II.

3.3. The variational case

3.3.1. Main result

In this Subsection we assume that

$$\mu \in W^{-1}E_{\overline{M}}(\Omega).$$

Consider the following unilateral problem

$$\begin{cases} u\in K_\psi,\ g(x,u,\nabla u)\in L^1(\Omega),\ g(x,u,\nabla u)u\in L^1(\Omega)\\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla(u-v)\,dx + \int_\Omega g(x,u,\nabla u)(u-v)\,dx\\ \qquad\qquad\le\langle\mu,u-v\rangle,\\ \forall\ \ v\in K_\psi\cap L^\infty(\Omega).\end{cases}\tag{3.3.1}$$

Our main result is then the following:

Theorem 3.3.1. *Under the assumptions* (A_1) - (A_5), (G_1) *and* (G_2), *there exists at least one solution of (3.3.1).*

Remark 3.3.1. *We remark that the statement of the previous Theorem does not exists in the case of Sobolev space. But, some existence result in this sense have been proved under the regularity assumption* $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *(see [44]).*

Remark 3.3.2. *We recall that, differently from the methods used in [44] and [74], we do not introdecing the function* ψ^+ *in the test function used in the step of a priori estimate.*

3.3.2. Approximate problems and a priori estimate

Let us define

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}$$

and let us consider the approximate unilateral problems:

$$\begin{cases} u_n \in K_\psi \cap \mathcal{D}(A), \\ \langle Au_n, u_n - v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \le \langle \mu, u_n - v\rangle, \\ \forall v \in K_\psi. \end{cases} \tag{3.3.2}$$

Applying Lemma 2.2.1, the problem (3.3.2) has at least one solution.

Proposition 3.3.1. *Assume that the assumption* $(A_1) - (A_5)$, G_1 *and* G_2 *hold, and let* u_n *be a solution of the approximate problem* (3.3.2). *Then there exists a constant c (which does not depend on the n) such that*

$$\int_\Omega M(|\nabla u_n|)\,dx \le c.$$

Proof of Proposition 3.3.1

STEP 1. we chow that $\int_\Omega M(|\nabla T_k(u_n)|)\,dx \le c(k)$.

Let $k \ge \|v_0\|_\infty$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = (\frac{2b(k)}{\alpha})^2$.
It is well know that

$$\varphi_k'(s) - \frac{2b(k)}{\alpha}|\varphi_k(s)| \ge \frac{1}{2}, \quad \forall s \in I\!R. \tag{3.3.3}$$

Since $\mu \in W^{-1}E_{\overline{M}}(\Omega)$ then μ can be write as follows

$$\mu = f_0 - \mathrm{div}F \text{ whith } f_0 \in E_{\overline{M}}(\Omega),\ F \in (E_{\overline{M}}(\Omega))^N.$$

Taking $u_n - \eta\varphi_k(T_l(u_n - v_0))$ as test function in (3.3.2), where $l = k + \|v_0\|_\infty$, we obtain,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0))\,dx$$
$$\le \int_\Omega f_0\varphi_k(T_l(u_n - v_0))\,dx + \int_\Omega F\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx.$$

Since $g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n-v_0))\geq 0$ on the subset $\{x\in\Omega : |u_n(x)|>k\}$, then

$$\begin{aligned}\int_{\{|u_n-v_0|\leq l\}} & a(x,u_n,\nabla u_n)\nabla(u_n-v_0)\varphi_k'(T_l(u_n-v_0))\,dx\\ &\leq \int_{\{|u_n|\leq k\}} |g_n(x,u_n,\nabla u_n)||\varphi_k(T_l(u_n-v_0))|\,dx+\int_\Omega f_0\varphi_k(T_l(u_n-v_0))\,dx\\ &+\int_{\{|u_n-v_0|\leq l\}} F\nabla u_n\varphi_k'(T_l(u_n-v_0))\,dx-\int_{\{|u_n-v_0|\leq l\}} F\nabla v_0\varphi_k'(T_l(u_n-v_0))\,dx.\end{aligned}$$

By using $(A_4),(G_1)$ and Young's inequality, we have

$$\begin{aligned}\alpha\int_{\{|u_n-v_0|\leq l\}} & M(|\nabla u_n|)\varphi_k'(T_l(u_n-v_0))\,dx\\ &\leq b(|k|)\int_\Omega (c(x)+M(|\nabla T_k(u_n)|))\,|\varphi_k(T_l(u_n-v_0))|\,dx\\ &+\int_\Omega \delta(x)\varphi_k'(T_l(u_n-v_0))\,dx+\int_\Omega f_0\varphi_k(T_l(u_n-v_0))\,dx\\ &+\tfrac{\alpha}{2}\int_{\{|u_n-v_0|\leq l\}} M(|\nabla u_n|)\varphi_k'(T_l(u_n-v_0))\,dx+C_1(k).\end{aligned}$$

Which implies that

$$\begin{aligned}\tfrac{\alpha}{2}\int_{\{|u_n-v_0|\leq l\}} & M(|\nabla u_n|)\varphi_k'(T_l(u_n-v_0))\,dx\\ &\leq b(|k|)\int_\Omega (c(x)+M(|\nabla T_k(u_n)|))\,|\varphi_k(T_l(u_n-v_0))|\,dx\\ &+\int_\Omega \delta(x)\varphi_k'(T_l(u_n-v_0))\,dx+\int_\Omega f_0\varphi_k(T_l(u_n-v_0))\,dx+C_1(k).\end{aligned}$$

Since $\{x\in\Omega, |u_n(x)|\leq k\}\subseteq\{x\in\Omega : |u_n-v_0|\leq l\}$ and the fact that h,δ and $f_0\in L^1(\Omega)$, then

$$\int_\Omega M(|\nabla T_k(u_n)|)\varphi_k'(T_l(u_n-v_0))\,dx\leq\frac{2b(k)}{\alpha}\int_\Omega M(|\nabla T_k(u_n)|)|\varphi_k(T_l(u_n-v_0))|\,dx+C_2(k).$$

Which implies that

$$\int_\Omega M(|\nabla T_k(u_n)|)\left[\varphi_k'(T_l(u_n-v_0))-\frac{2b(k)}{\alpha}|\varphi_k(T_l(u_n-v_0))|\right]\,dx\leq C_2(k).$$

By using (3.3.3), we deduce,

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx\leq C(k). \tag{3.3.4}$$

STEP 2. We prove that $\int_\Omega M(|\nabla u_n|)\,dx\leq C$.

Taking $v=v_0$ as test function in (3.3.2), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla v_0)\,dx &+\int_\Omega g_n(x,u_n,\nabla u_n)(u_n-v_0)\,dx\\ &\leq\int_\Omega f_0(u_n-v_0)\,dx+\int_\Omega F\nabla(u_n-v_0)\,dx.\end{aligned}$$

Let $k > \|v_0\|_\infty$, since $g_n(x,u_n,\nabla u_n)(u_n - v_0) \geq 0$ in the subset $\{x \in \Omega; |u_n(x)| \geq k\}$, we deduce

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)\,dx + \int_{\{|u_n(x)|\leq k\}} g_n(x,u_n,\nabla u_n)(u_n - v_0)\,dx \\ \leq \int_\Omega f_0(u_n - v_0)\,dx + \int_\Omega F\nabla u_n\,dx - \int_\Omega F\nabla v_0\,dx.$$

thus, implies that, by using (3.3.4) and (G_1)

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)\,dx \leq \int_\Omega f_0 u_n\,dx + \int_\Omega F\nabla u_n\,dx + C_4(k). \tag{3.3.5}$$

By using Lemma 1.1.3 of Chapter I and Young's inequality, we deduce

$$\int_\Omega f_0 u_n\,dx \leq C + \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)\,dx, \tag{3.3.6}$$

and

$$\int_\Omega F\nabla u_n\,dx \leq C' + \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)\,dx. \tag{3.3.7}$$

Combining (3.3.5), (3.3.6) and (3.3.7), we get

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n - \nabla v_0)\,dx \leq \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)\,dx + \frac{\alpha}{4}\int_\Omega M(|\nabla u_n|)\,dx + C_5(k),$$

which implies, by using (A_4)

$$\alpha\int_\Omega M(|\nabla u_n|)\,dx \leq \frac{\alpha}{2}\int_\Omega M(|\nabla u_n|)\,dx + C_6(k)$$

hence

$$\int_\Omega M(|\nabla u_n|)\,dx \leq C. \tag{3.3.8}$$

3.4. Modular convergence of truncation

Proposition 3.4.1. *Let u_n be a solutions of the problem* (3.3.2). *Then there exists a function $u \in W_0^1 L_M(\Omega)$ such that*

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \quad in \quad L^1(\Omega).$$

Proof of Proposition 3.4.1

STEP 1. Boundedeness of $(a(x,u_n,\nabla u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$.

The proof of this step is similarly to the one of the step 4 of Chapter II.

STEP 2. we claim that $M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|)$ **in** $L^1(\Omega)$.

In view of Proposition 3.2.1, u_n is bounded in $W_0^1 L_M(\Omega)$. So there exists some $u \in W_0^1 L_M(\Omega)$ such that

$$\begin{array}{l} u_n \rightharpoonup u \ \text{ weakly in } \ W_0^1 L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M, \Pi E_{\overline{M}}) \\ u_n \to u \text{ strongly in } \ E_M(\Omega) \ \text{ and } \ a.e. \ \text{ in } \ \Omega. \end{array} \tag{3.4.1}$$

Let $k > \|v_0\|_\infty$, for some $\eta > 0$ small enough, we take $u_n - \eta\varphi(T_k(u_n) - T_k(v_j))$ as function test in (3.3.2) and reasoning as in Chapter II, we get

$$\limsup_{s\to\infty} \limsup_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \, dx = 0,$$

then by using the Lemma 2.3.1, we deduce the result.

3.4.1. Proof of Theorem 3.3.1.

We can easy prove the following assertions
Assertion 1 There exists a constant c_1 such that

$$\int_\Omega |g_n(x, u_n, \nabla u_n)| \, dx \le c_1.$$

Assertion 2 There exists a constant c_2 such that

$$\int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx \le c_2.$$

Assertion 3 (Equi-integrability of $g_n(x, u_n, \nabla u_n)$).
The sequence $(g_n(x, u_n, \nabla u_n))_n$ is uniformly equi-integrability in Ω.
By applying the assertions described above and Proposition 3.4.1, we can easily pass to the limit in the approximate problem (3.3.2).

Chapter 4

Existence results for some unilateral problems without sign condition in Orlicz spaces[1]

We prove the existence results in the setting of Orlicz space for the unilateral problems associated to the following equation,

$$Au + g(x, u, \nabla u) = f,$$

where A is a Leray-Lions operator acting from its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ into its dual, while $g(x, u, \nabla u)$ is a nonlinear term having a growth condition with respect to ∇u and no growth with respect to u, but no satisfies any sign condition. The right hand side f belongs to $L^1(\Omega)$, and the obstacle is a measurable function.

4.1. Introduction

Let Ω be an open bounded subset of $I\!R^N$, with the segment property and let $f \in L^1(\Omega)$. Consider the following nonlinear Dirichlet problem :

$$Au + g(x, u, \nabla u) = f \tag{4.1.1}$$

where $Au = -\text{div}(a(x, u, \nabla u))$ is a Leray-Lions operator defined on $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$, with M an N-function and where g is a non-linearity with the following natural growth condition :

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|)), \tag{4.1.2}$$

and which satisfies the classical sign condition $g(x, s, \xi).\xi \geq 0$.
In the case where g depends only on x and u it is well known that Gossez and Mustoren [81] solved (4.1.1) with data are taken in $W^{-1}E_{\overline{M}}(\Omega)$.
In this context of nonlinear operators, if (4.1.2) holds true, the existence results for some strongly nonlinear equations (or unilateral) problem associated to (4.1.1) have been proved in [5, 39, 40] when f belongs either to $W^{-1}E_{\overline{M}}(\Omega)$ or $L^1(\Omega)$ but the results

[1]Journal Nonlinear Analysis Series A: Theory, Methods & Applications 68(2008)2362 – 2380.

is restricted to N-function M satisfing a Δ_2-condition.
In a recents works, the authors in [7, 8, 72, 73], studied the above results without assuming the Δ_2-condition.
In this latter works, the sign condition plays a principal role to obtain a priori estimates and existence of solutions.
Our purpose in this Chapter, is then to study the strongly nonlinear unilateral problems associated to the equation (4.1.1) but without assuming any sign condition and any regularity on the obstacle ψ. More precisely, we prove the existence result for the following unilateral problem

$$(P)\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega),\ u \geq \psi, \quad \text{a.e. in } \ \Omega,\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega (a(x,u,\nabla u))T_k(u-\varphi)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-\varphi)\,dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega fT_k(u-\varphi)\,dx \\ \text{for all } \varphi \in K_\psi \cap L^\infty(\Omega),\ \forall\ k>0. \end{cases}$$

where $f \in L^1(\Omega)$ and $K_\psi = \{v \in W_0^1L_M(\Omega), u \geq \psi, a.e. \ \text{in} \ \Omega\}$, with ψ a measurable function on Ω.
To overcome this difficulty (caused by the elimination the sign condition) in the present Chapter, we have changed the condition (4.1.1) by the following one

$$|g(x,s,\xi)| \leq c(x) + \rho(s)M(|\xi|),$$

the model problems is to consider $g(x,u,\nabla u) = c(x) + |\sin u|e^{-u^2}M(|\nabla|)$, where $c(x) \in L^1(\Omega)$.
In the classical Sobolev space $W_0^{1,p}(\Omega)$, Porretta in [103] have studied the problem (4.1.1) where the right hand side is a measure. Let us point out that another work in this direction can be found in [60] where the problem (4.1.1) is studied with $f \in L^m(\Omega)$ for that the authors have proved that there exists a bounded weak solution for $m > \frac{N}{2}$, and unbounded entropy solution for $\frac{N}{2} > m > \frac{2N}{N+2}$. A different approach (without used the sign condition) was introduced also in [59] when $b(x,s,\xi) = \lambda s - |\xi|^2$ with $\lambda > 0$.

4.2. Main results

Through this Chapter, Ω will be a bounded subset of $I\!R^N, N \geq 2$ with the segment property, M be an N-function and P be an N-function such that $P \ll M$.
Let K_ψ the convex set defined in (2.2.1) satisfies the conditions (A_5) and (A_7) (see Remark 2.3.2 of Chapter II).
We consider the Leray-Lions operator,

$$Au = -\text{div}(a(x,u,\nabla u)), \tag{4.2.1}$$

defined on $\mathcal{D}(A) \subset W_0^1L_M(\Omega)$ into $W^{-1}L_{\overline{M}}(\Omega)$ where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a function satisfies the conditions (A_1)-(A_3) and (A_6) of Chapter II.

Furthermore, let $g(x,s,\xi) : \Omega \times I\!R \times I\!R^N \to I\!R$ be a Carathéodory function such that for *a.e.* $x \in \Omega$ and for all $s \in I\!R,\ \xi \in I\!R^N$, the following growth condition,

$$|g(x,s,\xi)| \leq \gamma(x) + \rho(s)M(|\xi|), \tag{4.2.2}$$

is satisfied, where $\rho : I\!R \to I\!R^+$ is a continuous positive function belongs to $L^1(I\!R)$ and $\gamma(x)$ belongs to $L^1(\Omega)$.
We consider the following problem

$$(P)\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega),\ u \geq \psi \ \text{ a.e. in } \ \Omega,\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad \leq \displaystyle\int_\Omega fT_k(u-v)\,dx \\ \forall v \in K_\psi \cap L^\infty(\Omega) \ \text{ and } \ \forall k > 0. \end{cases}$$

The aim of this Chapter is to prove the following

Theorem 4.2.1. *Let $f \in L^1(\Omega)$. Assume that (A_1)-(A_3), (A_5)-(A_7) and (4.2.2) hold true. Then the problem (P) admits at least one solution.*

Remark 4.2.1. *The statement of Theorem* 4.2.1 *generalizes in Orlicz case the analogous one in [10].*

4.3 Proof of Theorem 4.2.1

4.3.1 Approximate problem and a priori estimates

Let f_n be a sequence of regular functions which strongly converges to f in $L^1(\Omega)$ such that $\|f_n\|_1 \leq \|f\|_1$.
Let us consider the approximate problem :

$$\begin{cases} u_n \in K_\psi \cap \mathcal{D}(A) \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f_n(u_n - v)\,dx \quad \forall v \in K_\psi(\Omega), \end{cases} \tag{4.3.1}$$

where

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|},$$

By Lemma 2.2.1, the approximate problem (4.3.1) has at least one solution.

4.3.2 A priori estimates

Proposition 4.3.1. *Assume that (A_1)-(A_3), (A_5)-(A_7) and (4.2.2) hold true and let $(u_n)_n$ be a solution of the approximate problem (4.3.1). Then for all $k > 0$, there exists a constant $c(k)$ (which does not depend on the n) such that*

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx \leq c(k) \tag{4.3.2}$$

Proof. Let $v_0 \in K_\psi \cap L^\infty(\Omega) \cap W_0^1 E_M(\Omega)$, such existence of v_0 is ensure by assumptions (A_5) and (A_7).

For η small enough, let $v = u_n - \eta \exp(G(u_n)) T_k(u_n - v_0)^+$ where $G(s) = \int_0^s \frac{\rho(t)}{\alpha} dt$ (the function ρ appears in (4.2.2)), choosing v as test function in (4.3.1) We have,

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \nabla(\exp(G(u_n)) T_k(u_n - v_0)^+) \, dx \\ &\qquad + \int_\Omega g_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx \\ &\qquad \leq \int_\Omega f_n \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx \end{aligned} \tag{4.3.3}$$

which gives

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_0)^+ \exp(G(u_n) \, dx \\ &\qquad + \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \frac{\rho(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx \\ &\qquad \leq \int_\Omega |g_n(x, u_n, \nabla u_n)| \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx \\ &\qquad + \int_\Omega f_n \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx, \end{aligned} \tag{4.3.4}$$

Moreover, from (4.2.2), one gets

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_0)^+ \exp(G(u_n) \, dx \\ &\qquad + \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \frac{\rho(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx \\ &\qquad \leq \int_\Omega \rho(u_n) M(|\nabla u_n|) \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx \\ &\qquad + \int_\Omega (f_n + \gamma(x)) \exp(G(u_n)) T_k(u_n - v_0)^+ \, dx, \end{aligned} \tag{4.3.5}$$

by using (A_6) and the fact that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \gamma \in L^1(\Omega)$,we have

$$\begin{aligned} &\int_{\{0 \leq u_n - v_0 \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) \, dx \\ &\qquad \leq \int_{\{0 \leq u_n - v_0 \leq k\}} a(x, u_n, \nabla u_n) \nabla v_0 \exp(G(u_n)) \, dx + c_1 \\ &\qquad \leq c \int_{\{0 \leq u_n - v_0 \leq k\}} a(x, u_n, \nabla u_n) \frac{\nabla v_0}{c} \exp(G(u_n)) \, dx + c_1 \end{aligned} \tag{4.3.6}$$

where c is a constant such that $0 < c < 1$,
and with (A_3), we deduce

$$\begin{aligned} &\int_{\{0 \leq u_n - v_0 \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) \, dx \\ &\qquad \leq c \Big\{ \int_{\{0 \leq u_n - v_0 \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) \, dx \\ &\qquad - \int_{\{0 \leq u_n - v_0 \leq k\}} a(x, u_n, \frac{\nabla v_0}{c})(\nabla u_n - \frac{\nabla v_0}{c}) \exp(G(u_n)) \, dx + c_1 \Big\} \end{aligned} \tag{4.3.7}$$

which implies that,

$$(1-c)\int_{\{0\le u_n-v_0\le k\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx$$
$$\le c\int_{\{0\le u_n-v_0\le k\}} |a(x,u_n,\frac{\nabla v_0}{c})||(\nabla u_n-\frac{\nabla v_0}{c})|\exp(G(u_n))\,dx+c_1$$
$$\le c\int_{\{0\le u_n-v_0\le k\}} |a(x,u_n,\frac{\nabla v_0}{c})||\frac{\nabla v_0}{c}|\exp(G(u_n))\,dx$$
$$+c\int_{\{0\le u_n-v_0\le k\}} |a(x,u_n,\frac{\nabla v_0}{c})||\nabla u_n|\exp(G(u_n))\,dx+c_1. \tag{4.3.8}$$

Since $\frac{\nabla v_0}{c}\in (E_M(\Omega))^N$, then by using the Young's inequality and the condition (A_2) we have

$$(1-c)\int_{\{0\le u_n-v_0\le k\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx \le \tfrac{\alpha(1-c)}{2}\int_{\{0\le u_n-v_0\le k\}} M(|\nabla u_n|)\exp(G(u_n))\,dx+c_2(k). \tag{4.3.9}$$

where $c_2(k)$ is a positive constant which depending only in k.
Finally, from (A_6), we can conclude

$$\int_{\{0\le u_n-v_0\le k\}} M(|\nabla u_n|)\exp(G(u_n))\,dx\le c_3(k).$$

Since $\exp(G(-\infty))\le\exp(G(u_n))\le\exp(G(+\infty))$ and $\exp(|G(\pm\infty)|)\le\exp(\frac{\|g\|_{L^1(\Omega)}}{\alpha})$, we get

$$\int_{\{0\le u_n-v_0\le k\}} M(|\nabla u_n|)\,dx\le c_4(k). \tag{4.3.10}$$

Similarly, taking $v=u_n+\exp(-G(u_n))T_k(u_n-v_0)^-$ as test function in (4.3.1), we obtain

$$(1-c)\int_{\{-k\le u_n-v_0\le 0\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(-G(u_n))\,dx \le \tfrac{\alpha(1-c)}{2}\int_{\{-k\le u_n-v_0\le 0\}} M(|\nabla u_n|)\exp(-G(u_n))\,dx+c_5(k), \tag{4.3.11}$$

and then

$$\int_{\{-k\le u_n-v_0\le 0\}} M(|\nabla u_n|)\,dx\le c_6(k). \tag{4.3.12}$$

Combining (4.3.10) and (4.3.12), we deduce

$$\int_{\{|u_n-v_0|\le k\}} M(|\nabla u_n|)\,dx\le c_7(k). \tag{4.3.13}$$

Since $\{x\in\Omega;|u_n(x)|\le k\}\subset\{x\in\Omega;|u_n-v_0|\le k+\|v_0\|_\infty\}$, we have

$$\int_\Omega M(|\nabla T_k(u_n)|)\,dx=\int_{\{|u_n|\le k\}} M(|\nabla u_n|)\,dx\le\int_{\{|u_n-v_0|\le k+\|v_0\|_\infty\}} M(|\nabla u_n|)\,dx.$$

Hence, the inequality (4.3.13) give the desired estimate (4.3.2).

Proposition 4.3.2. *Assume that (A_1)-(A_3), (A_5)-(A_7) and (4.2.2) hold true and let $(u_n)_n$ be a solution of the approximate problem (4.3.1). Then for all $k > h > \|v_0\|_\infty$, there exists a constant c (which does not depend on the n, k and h) such that*

$$\int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx \le ck \tag{4.3.14}$$

Proof. Let $k > h > \|v_0\|_\infty$. By using $v = u_n - \eta \exp(G(u_n))T_k(u_n - T_h(u_n))^+$ as test function in (4.3.1), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n - T_h(u_n))^+ \exp(G(u_n))\,dx \\ &\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{\rho(u_n)}{\alpha}\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx \\ &\qquad \le \int_\Omega |g_n(x,u_n,\nabla u_n)| \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx \\ &\qquad + \int_\Omega f_n \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx,\end{aligned} \tag{4.3.15}$$

which yields, thanks (4.2.2),

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n - T_h(u_n))^+ \exp(G(u_n))\,dx \\ &\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{\rho(u_n)}{\alpha}\exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx \\ &\qquad \le \int_\Omega \rho(u_n)M(|\nabla u_n|) \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx \\ &\qquad + \int_\Omega (f_n + \gamma(x)) \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx.\end{aligned} \tag{4.3.16}$$

which gives, by using (A_6)

$$\begin{aligned}&\int_{\{h \le u_n \le h+k\}} a(x,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))\,dx \\ &\qquad \le \int_\Omega (f_n + \gamma(x)) \exp(G(u_n))T_k(u_n - T_h(u_n))^+\,dx \le kc_7,\end{aligned}$$

hence

$$\int_{\{h \le u_n \le h+k\}} M(|\nabla u_n|)\,dx \le kc_8, \tag{4.3.17}$$

where c_8 is a positive constant not depending on n, k and h.
On the other side, consider the test function $v = u_n + \exp(-G(u_n))T_k(u_n - T_h(u_n))^-$ in (4.3.1) and reasoning as in (4.3.17), we get

$$\int_{\{-k-h \le u_n \le -h\}} M(|\nabla u_n|)\,dx \le kc_9. \tag{4.3.18}$$

From the inequalities (4.3.17), (4.3.18) follows the estimate (4.3.14).

Proposition 4.3.3. *Assume that (A_1)-(A_3), (A_5)-(A_7) and (4.2.2) hold true and let $(u_n)_n$ be a solution of the approximate problem (4.3.1). Then there exists a measurable function u such that for all $k > 0$ we have (for a subsequence still denote by u_n),* *1)* $u_n \to u$ *a.e. in* Ω*,*

2) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^1L_M(\Omega)$ *for* $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$*,*

3) $T_k(u_n) \to T_k(u)$ *strongly in* $E_M(\Omega)$ *and a.e. in* Ω*.*

Proof. Let $k > h > \|v_0\|_\infty$ large enough. Thanks to Lemma 1.1.3, there exist two positive constants C_{11} and C_{12} such that,

$$\int_\Omega M(C_{11}|T_k(u_n - T_h(u_n))|)\,dx \leq C_{12}\int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx, \tag{4.3.19}$$

Then, we deduce by using Proposition 4.3.2 that,

$$\begin{aligned} M(C_{11}k)\text{meas}\{|u_n - T_h(u_n)| > k\} &= \int_{\{|u_n - T_h(u_n)|>k\}} M(C_{11}|T_k(u_n - T_h(u_n))|)\,dx \\ &\leq C_{12}\int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx \leq kC_{13}. \end{aligned}$$

Hence,

$$\text{meas}(\{|u_n - T_h(u_n)| > k\}) \leq \frac{kC_{13}}{M(kC_{11})} \quad \text{for all } \ n \ \text{ and } \quad \text{for all } \ k > h > \|v_0\|_\infty. \tag{4.3.20}$$

Finally, we have

$$\text{meas}\{|u_n| > k\} \leq \text{meas}\{|u_n - T_h(u_n)| > k - h\} \leq \frac{(k-h)C_{13}}{M((k-h)C_{11})} \quad \text{ for all } \ n.$$

So, as in step 3 of Chapter II, we can easily completes the proof.

Proposition 4.3.4. *Assume that* (A_1)*-*(A_3)*,* (A_5)*-*(A_7) *and (4.2.2) hold true and let* $(u_n)_n$ *be a solution of the approximate problem (4.3.1). Then for all* $k > 0$*,*

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \ \textit{ strongly in } \ L^1(\Omega).$$

Proof. Following the same techniques used in step 4 of Chapter II we can easily prove that the sequence $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.
Now, we introduce the following function of one real variable s, which is defined as,

$$\begin{cases} h_j(s) = 1 & \text{If} \quad |s| \leq j \\ h_j(s) = 0 & \text{If} \quad |s| \geq j+1 \\ h_j(s) = j+1-s & \text{If} \quad j \leq s \leq j+1 \\ h_j(s) = s+j+1 & \text{If} \quad -j-1 \leq s \leq -j \end{cases} \tag{4.3.21}$$

with j a nonnegative real parameter.
Let $\Omega_s = \{x \in \Omega, |\nabla T_k(u(x))| \leq s\}$ and denote by χ_s the caracteristic function of Ω_s.
In order to prove the modular convergence of truncation $T_k(u_n)$, we shall show the

following assertions:
Assertion (i)

$$\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{j<|u_n|<j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{4.3.22}$$

Assertion (ii)

$$T_k(u_n) \longrightarrow T_k(u) \text{ modular convergence in } W_0^1L_M(\Omega). \tag{4.3.23}$$

Proof of assertion (i). If we take $v = u_n + \exp(-G(u_n))T_1(u_n - T_j(u_n))^-$ as test function in (4.3.1), we get,

$$\begin{aligned}\int_{\{-(j+1)<u_n<-j\}} & a(x,u_n,\nabla u_n)\nabla u_n \exp(-G(u_n))\,dx \\ & \leq \int_\Omega (-f_n+\gamma(x))\exp(-G(u_n))T_1(u_n - T_j(u_n))^-\,dx.\end{aligned} \tag{4.3.24}$$

Using the fact that

$$\exp(G(-\infty)) \leq \exp(-G(u_n)) \leq \exp(G(+\infty))$$

we deduce

$$\begin{aligned}\int_{\{-(j+1)<u_n<-j\}} & a(x,u_n,\nabla u_n)\nabla u_n\,dx \\ & \leq -c_{17}\int_\Omega (f_n(x)-\gamma(x))\exp(-G(u_n))T_1(u_n - T_j(u_n))^-\,dx,\end{aligned} \tag{4.3.25}$$

Since $f_n \to f$ in $L^1(\Omega)$ and $|f_n \exp(-G(u_n))T_1(u_n - T_j(u_n))^-| \leq \exp(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha})|f_n|$ then Vitali's Theorem, permits to confirm that,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega f_n \exp(-G(u_n))T_1(u_n - T_j(u_n))^-\,dx = 0. \tag{4.3.26}$$

Similarly, since $\gamma \in L^1(\Omega)$, we obtain,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega \gamma \exp(-G(u_n))T_1(u_n - T_j(u_n))^-\,dx = 0. \tag{4.3.27}$$

Together (4.3.25), (4.3.26) and (4.3.27), we conclude that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{-(j+1)<u_n<-j\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{4.3.28}$$

On the other hand, taking $v = u_n - \eta\exp(G(u_n))T_1(u_n - T_j(u_n))^+$ as test function in (4.3.1) and reasoning as in the proof of (4.3.28), we deduce that

$$\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{j<u_n<j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{4.3.29}$$

Thus (4.3.22) follows from (4.3.28) and (4.3.29).

Proof of assertion (ii). Let $k \geq \|v_0\|_\infty$. By using (A_5) there exists a sequence $v_i \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_M(\Omega)$.

Let us $v = u_n - \eta \exp(G(u_n))(T_k(u_n) - T_k(v_i))^+ h_j(u_n)$ as test function in (4.3.1), we obtain by using (A_6) and (4.2.2)

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} & \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(v_i))h_j(u_n)\,dx \\ & - \int_{\{j<u_n<j+1\}} \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-T_k(v_i))^+\,dx \\ & \leq \int_\Omega \gamma(x)(T_k(u_n)-T_k(v_i))^+ h_j(u_n)\exp(G(u_n))\,dx \\ & + \int_\Omega f_n(T_k(u_n)-T_k(v_i))^+ h_j(u_n)\exp(G(u_n))\,dx.\end{aligned}$$

Thanks to (4.3.29), the second integral tend to zero as n and j tend to infinity, and by Lebesgue Theorem, we deduce that the right hand side converge to zero as n and i goes to infinity.

Then the least inequality becomes,

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} & \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla((T_k(u_n)-T_k(v_i))h_j(u_n)\,dx \\ & - \int_{\{T_k(u_n)-T_k(v_i)\geq 0, |u_n|>k\}} \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla T_k(v_i)h_j(u_n)\,dx \leq \varepsilon(n,i,j).\end{aligned} \tag{4.3.30}$$

Now, observe that

$$\begin{aligned}\left|\int_{\{T_k(u_n)-T_k(v_i)\geq 0,\ |u_n|>k\}} \exp(G(u_n))a(x,u_n,\nabla u_n)\nabla T_k(v_i)h_j(u_n)\,dx\right| \\ \leq c\int_{\{|u_n|>k\}} |a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))||\nabla v_i|\,dx.\end{aligned} \tag{4.3.31}$$

On the one hand, since $(|a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))|)_n$ is boundeed in $L_{\overline{M}}(\Omega)$, we get for a subsequence, $|a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))| \rightharpoonup \delta_j$ weakly in $L_{\overline{M}}(\Omega)$ for $\sigma(L_{\overline{M}}(\Omega), E_M(\Omega))$ for some $\delta_j \in L_{\overline{M}}(\Omega)$ and since $|\nabla v_i|\chi_{\{|u_n|>k\}}$ converges strongly to $|\nabla v_i|\chi_{\{|u|>k\}}$ in $E_M(\Omega)$ we have by letting $n \to \infty$

$$\begin{aligned}\int_{\{|u_n|>k\}} & |a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))||\nabla v_i|\,dx \\ & \to \int_{\{|u|>k\}} \delta_j|\nabla v_i|\,dx.\end{aligned}$$

Using now, the modular converge of $(v_i)_i$, we get

$$\int_{\{|u|>k\}} \delta_j|\nabla v_i|\,dx \to \int_{\{|u|>k\}} \delta_j|\nabla T_k(u)|\,dx.$$

Since $\nabla T_k(u) = 0$ in $\{|u|>k\}$ we deduce that

$$\int_{\{|u_n|>k\}} |a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))||\nabla v_i|\,dx = \varepsilon(n,i,j)$$

Combining this with (4.3.30) and (4.3.31)

$$\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(v_i))h_j(u_n)\,dx \leq \epsilon(n,i,j). \tag{4.3.32}$$

On the other side, we have

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(v_i))h_j(u_n)\,dx \\
&\quad\geq \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)] \\
&\qquad\qquad \times[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\,dx \\
&\quad + \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),T_k(v_i)\chi_s^i)[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\,dx \\
&\qquad\qquad -c\int_{\Omega\setminus\Omega_s^i} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla v_i|\,dx.
\end{aligned} \tag{4.3.33}$$

Where χ_s^i denotes the characteristics function of the subset

$$\Omega_s^i = \{x\in\Omega : \quad |\nabla T_k(v_i)| \leq s\}.$$

and as above

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_s^i} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla v_i|\,dx \\
&\qquad = \int_{\Omega\setminus\Omega_s} \delta_k|\nabla T_k(u)|\,dx + \varepsilon(n,i,j)
\end{aligned} \tag{4.3.34}$$

For what concerns the second term of the right hand side of the (4.3.33) we can write,

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\,dx \\
&\qquad = \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(T_k(u_n)))a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)\nabla T_k(u_n)\,dx \\
&\qquad - \int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)\nabla T_k(v_i)\chi_s^i h_j(u_n)\,dx
\end{aligned} \tag{4.3.35}$$

Starting with the first term of the last equality, we have by letting $n\to\infty$

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(T_k(u_n)))a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)\nabla T_k(u_n)\,dx \\
&\qquad = \int_{\{T_k(u)-T_k(v_i)\geq 0\}} \exp(G(T_k(u)))a(x,T_k(u),\nabla T_k(v_i)\chi_s^i)\nabla T_k(u)\,dx + \varepsilon(n),
\end{aligned}$$

since

$$\begin{aligned}
&\exp(G(T_k(u_n)))a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)\chi_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \\
&\qquad \longrightarrow \exp(G(T_k(u)))a(x,T_k(u),\nabla T_k(v_i)\chi_s^i)\chi_{\{T_k(u)-T_k(v_i)\geq 0\}}
\end{aligned}$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by using Lemma 1.1.6 while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ for $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$.

Letting again $i \to \infty$, one has, since
$a(x, T_k(u), \nabla T_k(v_i)\chi_s^i)\chi_{\{T_k(u)-T_k(v_i)\geq 0\}} \to a(x, T_k(u), \nabla T_k(u)\chi_s)$ strongly in $(E_{\overline{M}}(\Omega))^N$ by using the modular convergence of v_i and Lebesgue Theorem

$$\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(T_k(u_n)))a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)\nabla T_k(u_n)\, dx$$
$$= \int_\Omega \exp(G(u))a(x, T_k(u), \nabla T_k(u)\chi_s)\nabla T_k(u)\, dx + \varepsilon(n, i, j).$$

In the same way, we have

$$-\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)\nabla T_k(v_i)\chi_s^i h_j(u_n)\, dx$$
$$= -\int_\Omega \exp(G(u))a(x, T_k(u), \nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s\, dx + \varepsilon(n, i, j).$$

Adding the two equalities we get

$$\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} \exp(G(u_n))a(x, T_k(u_n), T_k(v_i)\chi_s^i)[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\, dx = \varepsilon(n, i, j). \tag{4.3.36}$$

Combining (4.3.32)-(4.3.34) and (4.3.36), we then conclude

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\geq 0\}} &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)]\\ &\times[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\, dx\\ &\leq c_{18}\int_{\Omega\setminus\Omega_s} \delta_k|\nabla T_k(u)|\, dx + \varepsilon(n, i, j)\end{aligned} \tag{4.3.37}$$

Now, taking $v = u_n + \exp(-G(u_n))(T_k(u_n) - T_k(v_i))^- h_j(u_n)$ as test function in (4.3.1) and reasoning as in (4.3.37) it is possible to conclude that

$$\begin{aligned}\int_{\{T_k(u_n)-T_k(v_i)\leq 0\}} &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)]\\ &\times[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\, dx\\ &\leq c_{19}\int_{\Omega\setminus\Omega_s} \delta_k|\nabla T_k(u)|\, dx + \varepsilon(n, i, j).\end{aligned} \tag{4.3.38}$$

Finally by using (4.3.37) and (4.3.38), we get

$$\begin{aligned}\int_\Omega &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_i)\chi_s^i)]\\ &\times[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\, dx\\ &\leq c_{20}\int_{\Omega\setminus\Omega_s} \delta_k|\nabla T_k(u)|\, dx + \varepsilon(n, i, j).\end{aligned} \tag{4.3.39}$$

On the other hand, we have

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]h_j(u_n)\,dx \\
&- \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)][\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\,dx \\
&\qquad = \int_\Omega a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\,dx \\
&\qquad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]h_j(u_n)\,dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_i)\chi_s^i - \nabla T_k(u)\chi_s]h_j(u_n)\,dx,
\end{aligned}
\tag{4.3.40}
$$

an, as it can be easily seen, each integral of the right-hand side of the form $\varepsilon(n,i,j)$ implying that

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]h_j(u_n)\,dx \\
&\qquad = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_i)\chi_s^i)] \\
&\qquad\quad \times[\nabla T_k(u_n) - \nabla T_k(v_i)\chi_s^i]h_j(u_n)\,dx + \epsilon(n,i,j).
\end{aligned}
\tag{4.3.41}
$$

Furthermore, using (4.3.39) and (4.3.41), we have

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]h_j(u_n)\,dx \\
&\qquad \le c_{21}\int_{\Omega\setminus\Omega_s} \delta_k|\nabla T_k(u)|\,dx + \varepsilon(n,i,j).
\end{aligned}
\tag{4.3.42}
$$

Now, we remark that

$$
\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx \\
&\qquad = \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)) \\
&\qquad\quad \times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)h_j(u_n)\,dx \\
&\qquad + \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)) \\
&\qquad\quad \times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)(1-h_j(u_n))\,dx.
\end{aligned}
\tag{4.3.43}
$$

Since $1-h_j(u_n)=0$ in $\{|u_n(x)|\le j\}$, then for j large enough the second term of the right hand side of (4.3.43) can write as follows

$$
\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)(1-h_j(u_n))\,dx \\
&\qquad = -\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s(1-h_j(u_n)\,dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u))\chi_s(1-h_j(u_n))\,dx.
\end{aligned}
\tag{4.3.44}
$$

Thanks to $(a(x,T_k(u_n),\nabla T_k(u_n)))$ is bounded in $(L_{\overline{M}}(\Omega))^N$ uniformly on n while $\nabla T_k(u)\chi_s(1-h_j(u_n))$ converge to zero strongly in $(L_M(\Omega))^N$ hence the first term

of the right hand side of (4.3.44) converge to zero as n goes to infinity.
The second term converges to zero because $\nabla T_k(u)\chi_s(1-h_j(u_n)) \rightharpoonup \nabla T_k(u)\chi_s(1-h_j(u)) = 0$ strongly in $E_M(\Omega)$ and By the continuity of the Nymetskii operator $a(x, T_k(u_n), \nabla T_k(u)\chi_s)$ converge strongly to $a(x, T_k(u), \nabla T_k(u)\chi_s)$. Finally, we deduce that

$$\lim_{n\to\infty}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)) \\ \times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)(1-h_j(u_n))\,dx = 0. \tag{4.3.45}$$

Combining (4.3.42), (4.3.43) and (4.3.45), we get

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \\ \leq c_{22}\int_{\Omega\setminus\Omega_s}\delta_k|\nabla T_k(u)|\,dx + \varepsilon(n,i,j). \tag{4.3.46}$$

Letting n, i, j and s to infinity, we deduce

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \to 0$$

as $n \to +\infty$ and $s \to +\infty$,
Thus implies by the Lemma 2.3.1 that

$$M(|\nabla T_k(u)|) \to M|\nabla T_k(u_n)| \quad \text{in} \quad L^1(\Omega). \tag{4.3.47}$$

4.3.3 Proof of Theorem 4.2.1

Step 1. Equi-integrability of the nonlinearities.
Thanks to (4.3.47), we obtain for a subsequence

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in} \quad \Omega.$$

Now, we show that :

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \;\text{ strongly in }\; L^1(\Omega). \tag{4.3.48}$$

On the one hand, let $v = u_n + \exp(-G(u_n))\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds$. Since $v \in W_0^1L_M(\Omega)$ and $v \geq \psi$, v is an admissible test function in (4.3.1). Then,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(-\exp(-G(u_n))\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds)\,dx \\ +\int_\Omega g_n(x,u_n,\nabla u_n)(-\exp(-G(u_n))\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds)\,dx \\ \leq \int_\Omega f_n(-\exp(-G(u_n))\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds)\,dx.$$

Which implies that, by using (4.2.2)

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{\rho(u_n)}{\alpha}\exp(-G(u_n))\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}\,ds\,dx\\ +\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\exp(-G(u_n))\rho(u_n)\chi_{\{u_n<-h\}}\,dx\\ \leq\int_\Omega\gamma(x)\exp(-G(u_n))\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}\,ds\,dx\\ +\int_\Omega\rho(u_n)M(|\nabla u_n|)\exp(-G(u_n))\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}\,ds\,dx\\ -\int_\Omega f_n\exp(-G(u_n))\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}\,ds\,dx\end{aligned}$$

using (A_6) and since $\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}\,ds\leq\int_{-\infty}^{-h}\rho(s)\,ds$, we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\exp(-G(u_n))\rho(u_n)\chi_{\{u_n<-h\}}\,dx\\ \leq\exp(\tfrac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha})\int_{-\infty}^{-h}\rho(s)\,ds(\|\gamma\|_{L^1(\Omega)}+\|f_n\|_{L^1(\Omega)})\\ \leq\exp(\tfrac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha})\int_{-\infty}^{-h}\rho(s)\,ds(\|\gamma\|_{L^1(\Omega)}+\|f\|_{L^1(\Omega)})\end{aligned}$$

using again (A_6), we obtain

$$\int_{\{u_n<-h\}}\rho(u_n)M(|\nabla u_n|)\,dx\leq c_{23}\int_{-\infty}^{-h}\rho(s)\,ds$$

and since $\rho\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}}\rho(u_n)M(|\nabla u_n|)\,dx=0. \tag{4.3.49}$$

On the other hand, let $M=\exp(\|\rho\|_{L^1(\Omega)})\int_0^{+\infty}\rho(s)\,ds$ and $h\geq M+\|v_0\|_{L^\infty(\Omega)}$. Consider $v=u_n-\exp(G(u_n))\int_0^{u_n}\rho(s)\chi_{\{s>h\}}\,ds$. Since $v\in W_0^1L_M(\Omega)$ and $v\geq\psi$, v is an admissible test function in (4.3.1). Then, similarly to (4.3.49), we deduce that

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}\rho(u_n)M(|\nabla u_n|)\,dx=0. \tag{4.3.50}$$

Combining (4.3.47), (4.3.49), (4.3.50) and Vitali's Theorem, we conclude (4.3.48).
Step 2. Passing to the limit
Let $v\in K_\psi\cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$, we take $u_n-T_k(u_n-v)$ as test function in (4.3.1), we can write

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\\ \leq\int_\Omega f_nT_k(u_n-v)\,dx.\end{aligned} \tag{4.3.51}$$

which implies that

$$\begin{aligned}\int_{\{|u_n-v|\le k\}} & a(x,u_n,\nabla u_n)\nabla u_n\,dx\\ &+\int_{\{|u_n-v|\le k\}} a(x,T_{k+\|v\|_\infty}u_n,\nabla T_{k+\|v\|_\infty}(u_n))\nabla v\,dx\\ &\qquad+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\le\int_\Omega f_nT_k(u_n-v)\,dx.\end{aligned}\tag{4.3.52}$$

By Fatou's Lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$ on easily see that

$$\begin{aligned}\int_{\{|u-v|\le k\}} & a(x,u,\nabla u)\nabla u\,dx\\ &-\int_{\{|u-v|\le k\}} a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla v)\,dx\\ &\qquad+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx\le\int_\Omega fT_k(u-v)\,dx.\end{aligned}\tag{4.3.53}$$

Hence

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx&+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx\\ &\le\int_\Omega fT_k(u-v)\,dx.\end{aligned}\tag{4.3.54}$$

Now, let $v\in K_\psi\cap L^\infty(\Omega)$, by the condition (A_5) there exists $v_j\in K_\psi\cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$ such that v_j converges to v modular, let $h>\max(\|v_0\|_\infty,\|v\|_\infty)$, taking $v=T_h(v_j)$ in (4.3.54), we have

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(v_j))\,dx&+\int_\Omega g(x,u,\nabla u)T_k(u-T_h(v_j))\,dx\\ &\le\int_\Omega fT_k(u-T_h(v_j))\,dx.\end{aligned}\tag{4.3.55}$$

We can easily pass to the limit as $j\to+\infty$ to get

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(v))\,dx&+\int_\Omega g(x,u,\nabla u)T_k(u-T_h(v))\,dx\\ &\le\int_\Omega fT_k(u-T_h(v))\,dx\quad\forall v\in K_\psi\cap L^\infty(\Omega),\end{aligned}\tag{4.3.56}$$

and since $h\ge\max(\|v_0\|_\infty,\|v\|_\infty)$, we deduce

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx&+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx\\ &\le\int_\Omega fT_k(u-v)\,dx\quad\forall v\in K_\psi\cap L^\infty(\Omega),\quad\forall\ k>0.\end{aligned}\tag{4.3.57}$$

thus, the proof of the Theorem 4.2.1 is now completes.

Chapter 5

Existence of solutions for unilateral problems in L^1 involving lower order terms in divergence form in Orlicz spaces [1]

This Chapter is concerned with the existence result of the unilateral problem associated to the equations of the type

$$Au - \text{div}\phi(u) = f \in L^1(\Omega),$$

where A is a Leray-Lions operator having a growth not necessarily of polynomial type and $\phi \in C^0(I\!R, I\!R^N)$.

5.1. Introduction

Let Ω be a bounded open subset of $I\!R^N$, and let p be a real number with $1 < p < +\infty$. Consider the following nonlinear Dirichlet problem :

$$Au - \text{div}\phi(u) = f, \tag{5.1.1}$$

where $Au = -\text{div } a(x, u, \nabla u)$ is a Leray-Lions operators defined from $W_0^{1,p}(\Omega)$ into its dual and ϕ lies in $C^0(I\!R, I\!R^N)$.
Boccardo proved in [45] the existence of entropy solution for the problem (5.1.1). The formulation adequate in this case is the following,

$$\begin{cases} u \in W_0^{1,q}(\Omega), \ \forall q < \frac{N(p-1)}{N-1} \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega \phi(u)\nabla T_k(u - v)\, dx \leq \int_\Omega f T_k(u - v)\, dx \\ \forall \ \ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

[1] This Chapter is the subject of two articles published respectively in
1) Journal of Applied Analysis, vol. 13, N.2(2007), 151 − 181.
2) Applicationes Mathematicae, 33, 2(2006), 217 − 241.

In this direction, Boccardo and Cirmi [46] are studied the existence and uniqueness of solution of the following unilateral problem,

$$\begin{cases} u \in W_0^{1,q}(\Omega),\ \forall q < \frac{N(p-1)}{N-1}, u \geq \psi \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\ dx \leq \int_\Omega f T_k(u-v)\ dx \\ \forall\ \ v \in K_\psi \cap L^\infty(\Omega), \end{cases}$$

where $K_\psi = \Big\{u \in W_0^{1,p}(\Omega) : u \geq \psi\Big\}$, with a measurable function $\psi : \Omega \to \overline{I\!R}$ such that $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In these results the function $a(.)$ is supposed to satsfy a polynomial growth conditions with respect to u and ∇u.
In the case where $a(.)$ satisfies a more general growth condition with respect to u and ∇u (such growth to relax the coefficients of the operator A), the adequate space in which (5.1.1) can be studied is the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ where the N-function M is related to the actual growth of a. The solvability of (5.1.1) in this setting is studied by Gossez-Mustonen [81] in the variational case for $\phi = 0$. The case where f belongs to $L^1(\Omega)$ and $\phi = 0$ is treated in [38]. this last result is restricted to the N-functions which satisfy the Δ_2-condition (this condition appears in the boundedness of the term $\nabla T_k(u_n)$ in $L_M(\Omega)$, see p. 96-97 of [38]). More precisely, the authors have proved in the previous work existence and uniqueness of the following unilateral problem

$$\begin{cases} u \in W_0^1L_Q(\Omega),\ \forall Q \in \mathcal{A}_M \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\ dx \leq \int_\Omega f T_k(u-v)\ dx \\ \forall\ \ v \in K_\psi \cap L^\infty(\Omega), \end{cases}$$

where

$$\mathcal{A}_M = \Big\{Q : Q\ \ N-\text{function}\ \ : \tfrac{Q"}{Q'} \leq \tfrac{M"}{M'} \\ \text{and}\ \ \int_0^1 Q \circ H^{-1}(\frac{1}{t^{1-\frac{1}{N}}})\ dt < \infty\ \ \text{where}\ \ H(t) = \frac{M(t)}{t}\Big\}$$

and where $K_\psi = \{u \in W_0^1L_M(\Omega) : u \geq \psi\}$, with the following restrictions on the obstacle ψ

$$\psi \in W_0^1L_M(\Omega) \cap L^\infty(\Omega), \tag{5.1.2}$$

$$\text{there exists}\ \ \overline{\psi} \in K_\psi\ \ \text{such that}\ \ \psi - \overline{\psi}\ \ \text{is continuous on}\ \ \Omega. \tag{5.1.3}$$

While the case $\phi \neq 0$, is studied by Benkirane and Bennouna in [36] where an entropy solution for equation (5.1.1) is proved without assuming the Δ_2-condition.
Our purpose in this paper is to prove the existence of solutions for obstacle problem associated to (5.1.1) for general N-functions M.

5.2. Statement of main results

5.2.1. Basic assumptions

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$, with the segment property.
Given an obstacle function $\psi : \Omega \to \overline{I\!R}$, such that the convex set K_ψ defined in (2.2.1)

satisfies the condition (A_5) of Chapter II.
Let $Au = -\text{div}(a(x,u,\nabla u))$ be a Leray-Lions operators defined on its domain $\mathcal{D}(A) \subset W_0^1L_M(\Omega)$ into $W^{-1}E_{\overline{M}}(\Omega)$ where $a(x,s,\xi) : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfies the conditions (A_1)-(A_4) (see Chapter II).
Finally, we suppose that,

$$f \in L^1(\Omega), \tag{5.2.1}$$

$$\phi \in C^0(I\!R, I\!R^N), \tag{5.2.2}$$

and

$$K_\psi \cap L^\infty(\Omega) \neq \emptyset. \tag{5.2.3}$$

5.2.2. Principal results

The first result in this Chapter is concerned with $a \equiv a(x,\xi)$

Remark 5.2.1. *Giving some comparisons of our hypotheses and those of [38, 36] :*

1) *In [38], the authors have supposed the Δ_2-condition and hypotheses (5.1.3) which is stronger than our hypotheses (A_5) (see Remark 2.2.1).*
2) *When $\psi = -\infty$, the convex set K_ψ coincides with the space $W_0^1L_M(\Omega)$, this implies that (A_5) is verified. For that the authors in [36] have not need to (A_5).*

Remark 5.2.2. *Remark that, if we suppose that $a(x,\xi)\xi \geq \alpha M(|\xi|)$, then the hypotheses (A_4) is verified for all $v_0 \in K_\psi \cap W_0^1E_M(\Omega)$.*

Proof. Let $v_0 \in K_\psi \cap W_0^1E_M(\Omega)$ and let $\lambda > 0$ large enough, we have

$$a(x,\xi)(\xi - \nabla v_0) = a(x,\xi)\xi - \frac{1}{\lambda}a(x,\xi)(\lambda\nabla v_0). \tag{5.2.4}$$

On the other hand, by using (A_3), we have

$$\begin{aligned} -\tfrac{1}{\lambda}a(x,\xi)(\lambda\nabla v_0) \;\geq\; & -\tfrac{1}{\lambda}a(x,\xi)\xi - a(x,\lambda\nabla v_0)\nabla v_0 \\ & -\frac{\alpha(1-\frac{1}{\lambda})}{2}\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}|\xi| \end{aligned} \tag{5.2.5}$$

using the Young's inequality, we deduce that

$$\begin{aligned} -\frac{\alpha(1-\frac{1}{\lambda})}{2}\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}|\xi| \geq & -\frac{\alpha(1-\frac{1}{\lambda})}{2}M(|\xi|) \\ & -\frac{\alpha(1-\frac{1}{\lambda})}{2}\overline{M}\left(\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}\right). \end{aligned} \tag{5.2.6}$$

Combining (5.2.4), (5.2.5) and (5.2.6), we get

$$\begin{aligned} a(x,\xi)(\xi - \nabla v_0) \geq & \; a(x,\xi)\xi - \tfrac{1}{\lambda}a(x,\xi)\xi \\ & -\frac{\alpha(1-\frac{1}{\lambda})}{2}M(|\xi|) - \gamma(x), \end{aligned} \tag{5.2.7}$$

where

$$\gamma(x) = \frac{\alpha(1-\frac{1}{\lambda})}{2}\overline{M}\left(\frac{|a(x,\lambda\nabla v_0)|}{\frac{\alpha(\lambda-1)}{2}}\right) + a(x,\lambda\nabla v_0)\nabla v_0.$$

Finally, by the hypotheses, we deduce

$$a(x,\xi)(\xi-\nabla v_0) \geq \frac{\alpha(1-\frac{1}{\lambda})}{2}M(|\xi|) - \gamma(x)$$

Theorem 5.2.1. *Assume that the hypothesis* $(A_1)-(A_5)$ *(where we omited the dependence of* a *in the variable* s *) and* $(5.2.1)-(5.2.3)$ *hold. Then there exists at least one solution of the following unilateral problem,*

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega), u \geq \psi \text{ a.e. in } \Omega. \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, \\ \forall \ v \in K_\psi \cap L^\infty(\Omega), \ \forall k > 0. \end{cases} \tag{5.2.8}$$

The next result deals with the case in which the function $a \equiv a(x,s,\xi)$ and the condition (A_4) is reduced to

(A_4') There exists two strictly positive constants α, ν such that

$$a(x,s,\zeta)\zeta \geq \alpha M(\frac{|\zeta|}{\nu})$$

for a.e. x in Ω and all $s \in I\!R$, $\zeta \in I\!R^N$.

Theorem 5.2.2. *Under the hypotheses of Theorem 5.2.1 with the condition* (A_4') *instead of* (A_4)*. Then there exists at least one solution of the following unilateral problem,*

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega), u \geq \psi \text{ a.e. in } \Omega. \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u-v)\,dx, \\ \forall \ v \in K_\psi \cap L^\infty(\Omega), \ \forall k > 0. \end{cases} \tag{5.2.9}$$

Remark 5.2.3. *Remark that, in the previous results, we can not replace* $K_\psi \cap L^\infty(\Omega)$ *by only* K_ψ*, since in general the integral* $\int_\Omega \phi(u)\nabla T_k(u-v)\,dx$ *may not have a meaning.*

Remark 5.2.4. *Note that, if we take* $M(t) = |t|^p$ *in the previous statements, then we obtain some existence result in the classical Sobolev spaces (which appears a new result).*

5.3. Proof of principal results

Without loss the generality we take $\nu = 1$ in the condition (A'_4).

5.3.1. Proof of Theorem 5.2.1

Approximate problem.

Let us defined the following sequence of the unilateral problems

$$\begin{cases} u_n \in K_\psi, \\ \langle Au_n, u_n - v\rangle + \displaystyle\int_\Omega \phi(T_n(u_n))\nabla(u_n - v)\,dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n(u_n - v)\,dx, \\ \forall v \in K_\psi. \end{cases} \tag{5.3.1}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. Applying Lemma 2.2.1 this approximate problem has at last one solution.

Some intermediates results.

Proposition 5.3.1. *Assume that* $(A_1)-(A_5)$, (5.2.1)-(5.2.3) *hold true and let* u_n *be a solution of the approximate problem (5.3.1). Then for all* $k > 0$, *there exists a constant* $c(k)$ *(which does not depend on the* n*) such that,*

$$\|T_k(u_n)\|_{W_0^1L_M(\Omega)} \leq c(k).$$

Proposition 5.3.2. *Assume that* $(A_1)-(A_5)$, (5.2.1)-(5.2.3) *hold true and let* u_n *be a solution of the approximate problem (5.3.1), then there exists a measurable function* u *such that, for all* $k > 0$ *we have,*

1) $u_n \to u$ *a.e. in* Ω,

2) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^1L_M(\Omega)$ *for* $\sigma(\Pi L_M, \Pi E_{\overline{M}})$,

3) $T_k(u_n) \to T_k(u)$ *strongly in* $E_M(\Omega)$ *and a.e. in* Ω.

Proposition 5.3.3. *Assume that* $(A_1)-(A_5)$, (5.2.1)-(5.2.3) *hold true and let* u_n *be a solution of the approximate problem (5.3.1). Then for all* $k > 0$,

1) $(a(x, \nabla T_k(u_n)))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$,

2) $M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|)$ *in* $L^1(\Omega)$.

Passing to the limit.

Let $v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$. Taking $u_n - T_k(u_n - v)$ as test function in (5.3.1), we can write, for n large enough ($n > k + \|v\|_\infty$),

$$\begin{aligned}\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - v)\, dx + \int_\Omega \phi(u_n)\nabla T_k(u_n - v)\, dx \\ \leq \int_\Omega f_n T_k(u_n - v)\, dx.\end{aligned} \tag{5.3.2}$$

Which implies that,

$$\begin{aligned}\int_{\{|u_n - v| \leq k\}} a(x, \nabla u_n)\nabla(u_n - v_0)\, dx + \int_{\{|u_n - v| \leq k\}} a(x, \nabla T_{k+\|v\|_\infty}(u_n))\nabla(v_0 - v)\, dx \\ + \int_\Omega \phi(u_n)\nabla T_k(u_n - v)\, dx \\ \leq \int_\Omega f_n T_k(u_n - v)\, dx.\end{aligned} \tag{5.3.3}$$

Applying the assertion 2) of Proposition 5.3.3, assertions 1), 3) of Proposition 5.3.2 and Fatou's Lemma, we have

$$\int_{\{|u - v| \leq k\}} a(x, \nabla u)\nabla(u - v_0)\, dx \leq \liminf_{n\to\infty} \int_{\{|u_n - v| \leq k\}} a(x, \nabla u_n)\nabla(u_n - v_0)\, dx, \tag{5.3.4}$$

on the other hand by Proposition 5.3.3 we get

$$a(x, \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, \nabla T_{k+\|v\|_\infty}(u)) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M),$$

which and assertion 1) of Proposition 5.3.2, Lebesgue's Theorem, allow to deduce

$$\int_{\{|u_n - v| \leq k\}} a(x, \nabla T_{k+\|v\|_\infty}(u_n))\nabla(v_0 - v)\, dx \to \int_{\{|u - v| \leq k\}} a(x, \nabla T_{k+\|v\|_\infty}(u))\nabla(v_0 - v)\, dx. \tag{5.3.5}$$

Moreover, thanks to assertion 1) and 2) of Proposition 5.3.2, we have

$$\int_\Omega \phi(u_n)\nabla T_k(u_n - v)\, dx \to \int_\Omega \phi(u)\nabla T_k(u - v)\, dx. \tag{5.3.6}$$

Combining (5.3.3)-(5.3.6), we get

$$\begin{aligned}\int_{\{|u - v| \leq k\}} a(x, \nabla u)\nabla(u - v_0)\, dx + \int_{\{|u - v| \leq k\}} a(x, \nabla T_{k+\|v\|_\infty}(u))\nabla(v_0 - v)\, dx \\ + \int_\Omega \phi(u)\nabla T_k(u - v)\, dx \leq \int_\Omega f T_k(u - v)\, dx.\end{aligned} \tag{5.3.7}$$

Hence

$$\begin{aligned}\int_\Omega a(x, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega \phi(u)\nabla T_k(u - v)\, dx \\ \leq \int_\Omega f T_k(u - v)\, dx.\end{aligned} \tag{5.3.8}$$

Now, let $v \in K_\psi \cap L^\infty(\Omega)$, by the condition (A_5) there exists $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v in modular sense. Let $h \geq \max(\|v_0\|_\infty, \|v\|_\infty)$, taking $v = T_h(v_j)$ in (5.3.8), we have

$$\begin{aligned}\int_\Omega a(x,\nabla u)\nabla T_k(u - T_h(v_j))\, dx &+ \int_\Omega \phi(u)\nabla T_k(u - T_h(v_j))\, dx \\ &\leq \int_\Omega f T_k(u - T_h(v_j))\, dx.\end{aligned} \tag{5.3.9}$$

We can easily pass to the limit as $j \to +\infty$ and get,

$$\begin{aligned}\int_\Omega a(x,\nabla u)\nabla T_k(u - T_h(v))\, dx &+ \int_\Omega \phi(u)\nabla T_k(u - T_h(v))\, dx \\ &\leq \int_\Omega f T_k(u - T_h(v))\, dx \quad \forall\, v \in K_\psi \cap L^\infty(\Omega).\end{aligned} \tag{5.3.10}$$

Finally, since $h \geq \max(\|v_0\|_\infty, \|v\|_\infty)$, we get

$$\begin{aligned}\int_\Omega a(x,\nabla u)\nabla T_k(u - v)\, dx &+ \int_\Omega \phi(u)\nabla T_k(u - v)\, dx \\ &\leq \int_\Omega f T_k(u - v)\, dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \forall k > 0.\end{aligned} \tag{5.3.11}$$

this, completes the proof of Theorem 5.2.1.

5.4. Proof of intermediates results

5.4.1. Proof of Proposition 5.3.1

Let $k > 0$. Taking $u_n - T_k(u_n - v_0)$ as test function in (5.3.1), we obtain for n large enough

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n - v_0)\, dx &+ \int_\Omega \phi(u_n)\nabla T_k(u_n - v_0)\, dx \\ &\leq \int_\Omega f_n T_k(u_n - v_0)\, dx.\end{aligned}$$

Since, $\nabla T_k(u_n - v)$ is identically zero on the set where $|u_n(x) - v_0(x)| > k$, hence we can write

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n - v_0)\, dx &\leq \int_{\{|u_n - v_0| \leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n|\, dx \\ &+ \int_{\{|u_n - v_0| \leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0|\, dx + \int_\Omega f_n T_k(u_n - v_0)\, dx.\end{aligned}$$

By using (A_4) and Young's inequality, we have

$$\begin{aligned}\alpha \int_{\{|u_n - v_0| \leq k\}} & M(|\nabla u_n|)\, dx \\ &\leq \tfrac{\alpha}{2} \int_{\{|u_n - v_0| \leq k\}} M(|\nabla u_n|)\, dx + C_0 \int_\Omega |\overline{M}(\phi(T_{k+\|v_0\|_\infty}(u_n))|)\, dx \\ &+ k\|f\|_{L^1} + \|\delta\|_{L^1}.\end{aligned}$$

Then, since $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ we can write,

$$\int_{\{|u_n - v_0| \leq k\}} M(|\nabla u_n|)\, dx \leq C_1(k),$$

where $c_1(k)$ is a constants which depends of k. Since k is arbitrary and $\{|u_n| \leq k\} \subset \{|u_n - v_0| \leq k + \|v_0\|_\infty\}$, we deduce that,

$$\int_\Omega M(|\nabla T_k(u_n)|)\, dx \leq \int_{\{|u_n - v_0| \leq k + \|v_0\|_\infty\}} M(|\nabla u_n|)\, dx \leq C_2(k). \qquad (5.4.1)$$

From which, we get

$$\|T_k(u_n)\|_{W_0^1 L_M(\Omega)} \leq C(k). \qquad (5.4.2)$$

5.4.2. Proof of Proposition 5.3.2

Let $k > h \geq \|v_0\|_\infty$. By using $v = u_n - T_k(u_n - T_h(u_n))$ as test function in (5.3.1) we obtain,

$$\int_\Omega a(x, \nabla u_n) \nabla T_k(u_n - T_h(u_n))\, dx + \int_\Omega \phi(T_n(u_n)) \nabla T_k(u_n - T_h(u_n))\, dx$$
$$\leq \int_\Omega f_n T_k(u_n - T_h(u_n))\, dx.$$

The second term of the left hand side of the last inequality vanishes for n large enough. Indeed. We have by virtue of Lemma 1.1.10,

$$\begin{aligned} \int_\Omega \phi(T_n(u_n)) \nabla T_k(u_n - T_h(u_n))\, dx &= \int_\Omega \phi(u_n) \nabla T_k(u_n - T_h(u_n))\, dx \\ &= \int_\Omega \mathrm{div} \left[\int_0^{u_n} \phi(s) \chi_{\{h \leq |s| \leq k+h\}}\, ds \right] dx = 0, \end{aligned}$$

(this is due to $\int_0^{u_n} \phi(s) \chi_{\{h \leq |s| \leq k+h\}}\, ds$ lies in $W_0^1 L_M(\Omega)$).
Thus,

$$\int_\Omega a(x, \nabla u_n) \nabla T_k(u_n - T_h(u_n))\, dx$$
$$\leq \int_\Omega f_n T_k(u_n - T_h(u_n))\, dx.$$

Which implies that,

$$\int_\Omega a(x, \nabla u_n) \nabla T_k(u_n - T_h(u_n))\, dx \leq k c_3, \qquad (5.4.3)$$

where c_3 is a nonnegative constant independent of n, k and h.
Now, let a constant c such that $0 < c < 1$ and satisfies $\frac{\alpha(1-c)}{2c} > \lambda > 1 + k_1$ (Such c is well existed since $\lim_{c \to 0^+} \dfrac{\alpha(1-c)}{2c} = +\infty$).
From (5.4.3) we have

$$\begin{aligned} &\int_\Omega a(x, \nabla T_k(u_n - T_h(u_n)))[\nabla T_k(u_n - T_h(u_n)) - (1-c)\nabla v_0]\, dx \\ &\qquad \leq c_3 k + \int_\Omega a(x, \nabla T_k(u_n - T_h(u_n)))(c-1)\nabla v_0\, dx \\ &\qquad = c_3 k + c \int_\Omega a(x, \nabla T_k(u_n - T_h(u_n)))\left(\frac{c-1}{c} \nabla v_0\right) dx \end{aligned}$$

and from the monotonicity condition (A_3) we get,

$$\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-(1-c)\nabla v_0]\,dx$$
$$\leq c_3k+c\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))\nabla T_k(u_n-T_h(u_n))\,dx$$
$$-c\int_\Omega a(x,\frac{c-1}{c}\nabla v_0)[\nabla T_k(u_n-T_h(u_n))-\frac{c-1}{c}\nabla v_0]\,dx.$$

Consequently,

$$(1-c)\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-\nabla v_0]\,dx$$
$$\leq c_3k+c_4+c\int_\Omega |a(x,\frac{c-1}{c}\nabla v_0)||\nabla T_k(u_n-T_h(u_n))|\,dx$$
$$=c_3k+c_4+\tfrac{\alpha(1-c)}{2}.\tfrac{2c}{\alpha(1-c)}\int_\Omega |a(x,\frac{c-1}{c}\nabla v_0)||\nabla T_k(u_n-T_h(u_n))|\,dx$$
$$=c_3k+c_4+\tfrac{\alpha(1-c)}{2}\int_\Omega \Big|\frac{a(x,\frac{c-1}{c}\nabla v_0)}{\frac{\alpha(1-c)}{2c}}\Big||\nabla T_k(u_n-T_h(u_n))|\,dx.$$

Thinks to the Young's inequality, we can deduce

$$(1-c)\int_\Omega a(x,\nabla T_k(u_n-T_h(u_n)))[\nabla T_k(u_n-T_h(u_n))-\nabla v_0]\,dx$$
$$\leq c_3k+c_4+\tfrac{\alpha(1-c)}{2}\int_\Omega \overline{M}(\frac{|a(x,\frac{c-1}{c}\nabla v_0)|}{\lambda})|\,dx$$
$$+\tfrac{\alpha(1-c)}{2}\int_\Omega M(|\nabla T_k(u_n-T_h(u_n))|)\,dx.$$

Finally, using (A_4), we deduce

$$\int_\Omega M(|\nabla T_k(u_n-T_h(u_n))|)\,dx\leq kC. \tag{5.4.4}$$

where C is a constant does not depends of n,k and h.
Furthermore, reasoning as in Proposition 4.3.3 of Chapter IV, we can easily deduce the result of Proposition 5.3.2.

5.4.3 Proof of proposition 5.3.3

1) Boundedness of $(a(x,\nabla T_k(u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$.
We fix $k>0$ and reasoning as in Chapter II, we deduce that $(a(x,\nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$. Which implies that, for all $k>0$, there exists a function $\rho_k\in(L_{\overline{M}}(\Omega))^N$ such that,

$$a(x,\nabla T_k(u_n))\rightharpoonup\rho_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}(\Omega),\Pi E_M(\Omega)). \tag{5.4.5}$$

2) We claim that $M(|\nabla T_k(u_n)|)\to M(|\nabla T_k(u)|)$ in $L^1(\Omega)$.
We fix $k>0$ and let $\Omega_r=\{x\in\Omega,|\nabla T_k(u(x))|\leq r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r\subset\Omega_{r+1}$ and $\mathrm{meas}(\Omega\backslash\Omega_r)\longrightarrow 0$ as $r\longrightarrow\infty$.

By using (A_5), there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_M(\Omega)$.
We will introduce the following function of one real variable s, which is defined as

$$h_m(s) = \begin{cases} 1 & \text{if} \quad |s| \leq m \\ -|s| + m + 1 & \text{if} \quad m \leq |s| \leq m+1 \\ 0 & \text{if} \quad |s| \geq m+1, \end{cases}$$

The choose of the $u_n - h_m(u_n - v_0)(T_k(u_n) - T_k(v_j))$ as test function in (5.3.1), we gives (using the fact that the derivative of $h_m(s)$ is different from zero only where $m < |s| < m+1$),

$$\begin{aligned} &\int_\Omega a(x, \nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j)) h_m(u_n - v_0)\, dx \\ &\qquad + \int_{\{m<|u_n - v_0|<m+1\}} a(x, \nabla u_n)\nabla(u_n - v_0)(T_k(u_n) - T_k(v_j)) h'_m(u_n - v_0)\, dx \\ &\qquad + \int_{\{m<|u_n - v_0|<m+1\}} \phi(u_n)\nabla(u_n - v_0)(T_k(u_n) - T_k(v_j)) h'_m(u_n - v_0)\, dx \\ &\qquad + \int_\Omega \phi(u_n)(\nabla T_k(u_n) - \nabla T_k(v_j)) h_m(u_n - v_0)\, dx \\ &\qquad \leq \int_\Omega f_n h_m(u_n - v_0)(T_k(u_n) - T_k(v_j))\, dx. \end{aligned} \tag{5.4.6}$$

We will deal with each term of (5.4.6). First of all, observe that

$$\int_\Omega f_n h_m(u_n - v_0)(T_k(u_n) - T_k(v_j))\, dx = \varepsilon(n, j). \tag{5.4.7}$$

Indeed. In view of assertion 1) of Proposition 5.3.2, we have

$$h_m(u_n - v_0)(T_k(u_n) - T_k(v_j)) \to h_m(u - v_0)(T_k(u) - T_k(v_j)) \text{ weakly}^* \text{ as } n \to +\infty \text{ in } Ł^\infty(\Omega),$$

and then,

$$\int_\Omega f_n h_m(u_n - v_0)(T_k(u_n) - T_k(v_j))\, dx \to \int_\Omega f h_m(u - v_0)(T_k(u) - T_k(v_j))\, dx \text{ as } n \to +\infty.$$

Since $h_m(u - v_0)(T_k(u) - T_k(v_j)) \to 0$ weak* in $L^\infty(\Omega)$ as $j \to +\infty$, we get

$$\int_\Omega f h_m(u - v_0)(T_k(u) - T_k(v_j))\, dx \to 0 \text{ as } j \to +\infty.$$

For what concerns the third term of the left hand side of (5.4.6), we have by letting $n \to \infty$

$$\begin{aligned} &\int_{\{m<|u_n - v_0|<m+1\}} \phi(u_n)\nabla(u_n - v_0)(T_k(u_n) - T_k(v_j)) h'_m(u_n - v_0)\, dx \\ &\qquad = \int_{\{m<|u - v_0|<m+1\}} \phi(u)\nabla(u - v_0)(T_k(u) - T_k(v_j)) h'_m(u - v_0)\, dx + \varepsilon(n) \end{aligned}$$

since $\phi(u_n)\chi_{\{m<|u_n - v_0|<m+1\}}(T_k(u_n) - T_k(v_j)) \to \phi(u)\chi_{\{m<|u - v_0|<m+1\}}(T_k(u) - T_k(v_j))$, strongly in $(E_{\overline{M}}(\Omega))^N$ by assertion 1) of Proposition 5.3.2 and Lebesgue Theorem while

$\nabla T_{m+1}(u_n) \rightharpoonup \nabla T_{m+1}(u_n)$ weakly in $(L_M(\Omega))^N$ by assertion 2) of Proposition 5.3.2. Letting $j \to \infty$ in the right term of the above equality, one has, by using the modular convergence of $(v_j)_j$

$$\int_{\{m<|u-v_0|<m+1\}} \phi(u)\nabla(u-v_0)(T_k(u)-T_k(v_j))h'_m(u-v_0)\,dx = \varepsilon(j)$$

and so

$$\int_{\{m<|u_n-v_0|<m+1\}} \phi(u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(v_j))h'_m(u_n-v_0)\,dx = \varepsilon(n,j). \quad (5.4.8)$$

Similarly, we have

$$\int_\Omega \phi(u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\,dx = \varepsilon(n,j). \quad (5.4.9)$$

Starting with the second term of the left hand side of (5.4.6), we have

$$\begin{aligned}
&\left|\int_{\{m<|u_n-v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(v_j))h'_m(u_n-v_0)\,dx\right| \\
&\qquad \le 2k\left|\int_{\{m<|u_n-v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x)\,dx\right| \\
&\qquad +2k\int_{\{m<|u_n-v_0|<m+1\}} \delta(x)\,dx
\end{aligned} \quad (5.4.10)$$

Moreover, since $\{m<|u_n-v_0|<m+1\} \subset \{l<|u_n|<l+s\}$ where $l = m-\|v_0\|_\infty$, $s = 2\|v_0\|_\infty + 1$, we get

$$\begin{aligned}
&2k\left|\int_{\{m<|u_n-v_0|<m+1\}} (a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x))\,\,dx\right| \\
&\qquad \le 2k\int_{\{l<|u_n|<l+s\}} (a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x))\,\,dx \\
&\qquad = 2k\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla u_n\,dx - 2k\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla v_0\,dx \\
&\qquad +4k\int_{\{l<|u_n|<l+s\}} \delta(x)\,dx,
\end{aligned} \quad (5.4.11)$$

Now, we take $u_n - T_s(u_n - T_l(u_n))$ as test function in (5.3.1), we get

$$\begin{aligned}
&\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla u_n\,dx + \int_\Omega \operatorname{div}\left[\int_0^{u_n} \phi(t)\chi_{\{l\le|t|\le l+s\}}\,dt\right]dx \\
&\qquad \le \int_\Omega f_n T_s(u_n-T_l(u_n))\,dx \le s\int_{\{|u_n|>l\}} |f_n|\,dx
\end{aligned}$$

and using the fact that $\int_0^{u_n} \phi(t)\chi_{\{l\le|t|\le l+s\}}\,dt \in W_0^1L_M(\Omega)$ and Lemma 1.1.10 one has,

$$\begin{aligned}
&\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla u_n\,dx \\
&\qquad \le \int_\Omega f_n T_s(u_n-T_l(u_n))\,dx \le s\int_{\{|u_n|>l\}} |f_n|\,dx.
\end{aligned} \quad (5.4.12)$$

On the other side, the Hölder's inequality gives

$$\left|-2k\int_{\{l<|u_n|<l+s\}} a(x,\nabla u_n)\nabla v_0\ dx\right| \leq 4k\|a(x,\nabla T_s(u_n-T_l(u_n)))\|_{\overline{M}}\|\nabla v_0\chi_{\{|u_n|>l\}}\|_M. \tag{5.4.13}$$

Furthermore, by the same argument as in the proof of step 3 of Chapter II, we get

$$\|a(x,\nabla T_s(u_n-T_l(u_n)))\|_{\overline{M}} \leq C_{14},$$

where C_{14} is a positive constant independent of n and m.
Combining (5.4.11), (5.4.12) and (5.4.13), we deduce

$$\begin{aligned}\left|2k\int_{\{m<|u_n-v_0|<m+1\}} (a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x))\ dx\right|\\ \leq C_{15}\int_{\{|u_n|>l\}}(\delta(x)+|f_n|)\ dx + C_{16}\|\nabla v_0\chi_{\{|u_n|>l\}}\|_M.\end{aligned} \tag{5.4.14}$$

Letting successively first n, then m $(l=m-\|v_0\|_\infty)$ go to infinity, we find, by using the fact that $\delta\in L^1(\Omega), v_0\in W_0^1E_M(\Omega)$ and the strong convergence of f_n

$$\left|\int_{\{m<|u_n-v_0|<m+1\}} (a(x,\nabla u_n)\nabla(u_n-v_0)+\delta(x))\ dx\right| = \varepsilon(n,m). \tag{5.4.15}$$

Finally, we have

$$\left|\int_{\{m<|u_n-v_0|<m+1\}} a(x,\nabla u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(v_j))h'_m(u_n-v_0)\ dx\right| = \varepsilon_j(n,m). \tag{5.4.16}$$

By means of (5.4.6)-(5.4.9), (5.4.16), we obtain

$$\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\ dx \leq \epsilon(n,m)+\epsilon(n,j). \tag{5.4.17}$$

Splitting the integral on the left hand side of (5.4.17) where $|u_n|\leq k$ and $|u_n|>k$, we can write,

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\ dx\\ &\qquad= \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]h_m(u_n-v_0)\ dx\\ &\qquad+\int_{\{|u_n|>k\}} a(x,0)\nabla T_k(v_j)h_m(u_n-v_0)\ dx\\ &\qquad-\int_{\{|u_n|>k\}} a(x,\nabla u_n)\nabla T_k(v_j)h_m(u_n-v_0)\ dx\\ &\qquad\geq \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]h_m(u_n-v_0)\ dx\\ &\qquad-\int_{\{|u_n|>k\}} |a(x,0)+a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))||\nabla v_j|\ dx.\end{aligned} \tag{5.4.18}$$

Since $(|a(x,0)+a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))|)_n$ is bounded in $L_{\overline{M}}(\Omega)$, we get, for a subsequence still denoted u_n

$$|a(x,0)+a(x,\nabla T_{m-\|v_0\|_\infty+1}(u_n))| \rightharpoonup l_m \quad \text{weakly in} \quad L_{\overline{M}}(\Omega) \quad \text{for} \quad \sigma(L_{\overline{M}},E_M),$$

and since, $|\nabla v_j|\chi_{\{|u_n|>k\}}$ converges strongly to $|\nabla v_j|\chi_{\{|u|>k\}}$ in $E_M(\Omega)$, we have by letting $n\to\infty$

$$-\int_{\{|u_n|>k\}} |a(x,0)+a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))|\nabla v_j|\,dx \to -\int_{\{|u|>k\}} l_m|\nabla v_j|\,dx$$

as n tends to infinity.
Using now, the modular convergence of $(v_j)_j$, we get

$$-\int_{\{|u|>k\}} l_m|\nabla v_j|\,dx \to -\int_{\{|u|>k\}} l_m|\nabla T_k(u)|\,dx$$

as j tends to infinity.
Since $\nabla T_k(u)=0$ in $\{|u|>k\}$, we deduce that,

$$-\int_{\{|u_n|>k\}} |a(x,0)+a(x,\nabla T_{m+\|v_0\|_\infty+1}(u_n))|\nabla v_j|\,dx = \varepsilon(n,j). \tag{5.4.19}$$

We then have by (5.4.18),

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)&(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\,dx\\ &\geq \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]h_m(u-v_0)\,dx+\epsilon(n,j).\end{aligned} \tag{5.4.20}$$

It is easily to see that,

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)&(\nabla T_k(u_n)-\nabla T_k(v_j))h_m(u_n-v_0)\,dx\\ &\geq \int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(v_j)\chi_s^j)]\\ &\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx\\ &\quad+\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx\\ &\quad-\int_{\Omega\setminus\Omega_s^j} |a(x,\nabla T_k(u_n))||\nabla v_j|\,dx+\epsilon(n,j),\end{aligned} \tag{5.4.21}$$

where χ_s^j denotes the characteristic function of the subset

$$\Omega_s^j=\{x\in\Omega : |\nabla T_k(v_j)|\leq s\},$$

and as above we have

$$-\int_{\Omega\setminus\Omega_s^j} |a(x,\nabla T_k(u_n))||\nabla v_j|\,dx = -\int_{\Omega\setminus\Omega_s} \rho_k|\nabla T_k(u)|\,dx+\varepsilon(n,j). \tag{5.4.22}$$

where ρ_k is some function in $L_{\overline{M}}(\Omega)$ such that

$$|a(x,\nabla T_k(u_n))| \rightharpoonup \rho_k \quad \text{weakly in} \quad L_{\overline{M}}(\Omega) \quad \text{for} \quad \sigma(L_{\overline{M}},E_M).$$

For what concerns the second term of the right hand side of (5.4.21) we can write,

$$\begin{aligned}\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx\\ =\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)h_m(T_k(u_n)-v_0)\,dx\\ -\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j h_m(u_n-v_0)\,dx.\end{aligned} \tag{5.4.23}$$

Starting of the second term of the last equality, we have

$$\begin{aligned}\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)h_m(u_n-v_0)\,dx\\ =\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u)h_m(u-v_0)\,dx+\varepsilon(n)\end{aligned}$$

since

$$a(x,\nabla T_k(v_j)\chi_s^j)h_m(T_k(u_n)-v_0)\to a(x,\nabla T_k(v_j)\chi_s^j)h_m(T_k(u)-v_0)$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by Lemma 1.1.6, while $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ for $\sigma(\prod L_M,\prod E_{\overline{M}})$.
Letting again $j\to\infty$, one has, since
$a(x,\nabla T_k(v_j)\chi_s^j)h_m(T_k(u)-v_0)\to a(x,\nabla T_k(u)\chi_s)h_m(T_k(u)-v_0)$ strongly in $(E_{\overline{M}}(\Omega))^N$ by using the modular convergence of v_j and Lebesgue Theorem

$$\begin{aligned}\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)h_m(u_n-v_0)\,dx\\ =\int_\Omega a(x,\nabla T_k(u)\chi_s)\nabla T_k(u)h_m(u-v_0)\,dx+\varepsilon(n,j).\end{aligned}$$

In the same way, we have

$$\begin{aligned}-\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j h_m(u_n-v_0)\,dx\\ =\int_{\Omega\setminus\Omega_s} a(x,\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s h_m(u-v_0)\,dx+\varepsilon(n,j).\end{aligned}$$

Adding the two equalities we conclude

$$\begin{aligned}\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx\\ =\int_{\Omega\setminus\Omega_s} a(x,0)\nabla T_k(u)h_m(u-v_0)\,dx+\varepsilon(n,j).\end{aligned}$$

Since $1-h_m(u-v_0)=0$ in $\{|u(x)-v_0(x)|\le m\}$ and since $\{|u(x)|\le k\}\subset\{|u(x)-v_0(x)|\le m\}$ for m large enough, we deduce

$$\begin{aligned}\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx\\ =\int_{\Omega\setminus\Omega_s} a(x,0)\nabla T_k(u)\,dx+\varepsilon(n,j).\end{aligned} \tag{5.4.24}$$

Combining (5.4.21), (5.4.22) and (5.4.24), we get

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(v_j)]h_m(u_n-v_0)\,dx&\\ \geq \int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(v_j)\chi_s^j)]&\\ \times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx&\\ -\int_{\Omega\setminus\Omega_s}\rho_k|\nabla T_k(u)|\,dx+\int_{\Omega\setminus\Omega_s}a(x,0)\nabla T_k(u)\,dx+\epsilon(n,j).&\end{aligned} \tag{5.4.25}$$

Which and (5.4.17)

$$\begin{aligned}\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(v_j)\chi_s^j)]&\\ \times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)]h_m(u_n-v_0)\,dx&\\ \leq \int_{\Omega\setminus\Omega_s}\rho_k|\nabla T_k(u)|\,dx+\int_{\Omega\setminus\Omega_s}a(x,0)\nabla T_k(u)\,dx&\\ +\epsilon(n,j)+\epsilon(n,m).&\end{aligned} \tag{5.4.26}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx&\\ -\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(v_j)\chi_s^j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx&\\ =\int_\Omega a(x,\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx&\\ -\int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx&\\ +\int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx,&\end{aligned} \tag{5.4.27}$$

an, as it can be easily seen that the term of the right-hand side is the form $\epsilon(n,j)$ implying that

$$\begin{aligned}\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]h_m(u_n-v_0)\,dx&\\ =\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(v_j)\chi_s^j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]h_m(u_n-v_0)\,dx+\epsilon(n,j).&\end{aligned} \tag{5.4.28}$$

Furthermore, using (5.4.26) and (5.4.28), we have

$$\begin{aligned}\int_\Omega [a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)]&\\ \times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s)]h_m(u_n-v_0)\,dx&\\ \leq \int_{\Omega\setminus\Omega_s}\rho_k|\nabla T_k(u)|\,dx+\int_{\Omega\setminus\Omega_s}a(x,0)\nabla T_k(u)\,dx&\\ +\epsilon(n,j)+\epsilon(n,m).&\end{aligned} \tag{5.4.29}$$

Now, we remark that

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \\
&\quad = \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n - v_0)\, dx \\
&\quad + \int_\Omega a(x,\nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_m(u_n - v_0))\, dx \\
&\quad - \int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s](1 - h_m(u_n - v_0))\, dx + \epsilon(n,j) + \epsilon(n,m).
\end{aligned} \tag{5.4.30}$$

Since $1 - h_m(u_n - v_0) = 0$ in $\{|u_n(x) - v_0(x)| \leq m\}$ and since $\{|u_n(x)| \leq k\} \subset \{|u_n(x) - v_0(x)| \leq m\}$ for m large enough, we deduce from (5.4.30)

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \\
&\quad = \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n - v_0)\, dx \\
&\quad - \int_{\{|u_n(x)|>k\}} a(x,0)\nabla T_k(u)\chi_s(1 - h_m(u_n - v_0))\, dx \\
&\quad + \int_{\{|u_n(x)|>k\}} a(x,\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s(1 - h_m(u_n - v_0))\, dx.
\end{aligned} \tag{5.4.31}$$

It is easy to see that, the two last terms of the last inequality tends to zero as $n \to \infty$, this implies that,

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \\
&\quad = \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] h_m(u_n - v_0)\, dx \\
&\qquad + \epsilon(n,j) + \epsilon(n,m).
\end{aligned} \tag{5.4.32}$$

Combining (5.4.17), (5.4.26), (5.4.29) and (5.4.32), we have

$$\begin{aligned}
&\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \\
&\qquad \leq \int_{\Omega\setminus\Omega_s} \rho_k \nabla T_k(u)\, dx + \int_{\Omega\setminus\Omega_s} a(x,0)\nabla T_k(u)\, dx + \epsilon(n,j,m).
\end{aligned} \tag{5.4.33}$$

By passing to the lim sup over n, and letting j, m, s tend to infinity, we obtain

$$\limsup_{s\to+\infty}\limsup_{n\to+\infty} \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx = 0.$$

Thus, by the Lemma 2.3.1, we get

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \ \text{ in } \ L^1(\Omega). \tag{5.4.34}$$

Remark 5.4.1. *If we assume that $\mathcal{A}_M \neq \emptyset$, then any solution of (5.2.8) belongs to $W_0^1 L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$.*

Proof Let $t \geq \|v_0\|_\infty$ and take $v = T_t(u)$ in (5.2.8), we get

$$\int_\Omega a(x, \nabla u) \nabla T_h(u - T_t(u))\, dx + \int_\Omega \phi(u) \nabla T_h(u - T_t(u))\, dx \\ \leq \int_\Omega f T_h(u - T_t(u))\, dx.$$

Hence,

$$\frac{1}{h} \int_\Omega a(x, \nabla u) \nabla T_h(u - T_t(u))\, dx \leq c.$$

Reasoning as above and letting $h \to 0$, we get

$$\lim_{h \to 0} \frac{1}{h} \int_{\{t \leq |u(x)| \leq t+h\}} M(|\nabla u|)\, dx \leq c.$$

Thus,

$$-\frac{d}{dt} \int_{\{|u(x)| > t\}} M(|\nabla u|)\, dx \leq c.$$

Following the same method used in the work of Benkirane and Bennouna [38] (see step 2 p. 93-97) one proves easily that $u \in W_0^1 L_Q(\Omega)\ \ \forall\, Q \in \mathcal{A}_M$.

In the case where $\psi = -\infty$ (i.e. $K_\psi = W_0^1 L_M(\Omega)$) it is possible to state :

Corollary 5.4.1. *Assume that* $(A_1) - (A_4)$ *and* (5.2.1), (5.2.2) *are satisfied. Then there exists at least one solution of the following problem*

$$\left\{ \begin{array}{l} u \in \mathcal{T}_0^{1,M}(\Omega), \\ \displaystyle\int_\Omega a(x, \nabla u) \nabla T_k(u - v)\, dx + \int_\Omega \phi(u) \nabla T_k(u - v)\, dx, \\ \qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u - v)\, dx, \\ \forall\ \ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{array} \right. \tag{5.4.35}$$

Remark 5.4.2. *Observe that the hypotheses* (A_5) *is not used in the previous corollary, this is due obviously to the density of* $\mathcal{D}(\Omega)$ *in* $W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ *in the modular sense (see [80]).*

Remark 5.4.3. *In the same particular case as above (i.e.* $\psi = -\infty$*), the element* v_0 *introduced in* (A_4) *lies in* $W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$*, then if we assume that* $\delta = v_0 = 0$ *and* $\mathcal{A}_M \neq \emptyset$*, then any solution of (5.4.35) belongs to* $W_0^1 L_Q(\Omega)$ *for each* $Q \in \mathcal{A}_M$*, which gives the result of [36].*

The proof is similar to that given in Remark 5.4.1.

Remark 5.4.4. *Let* $M(t) = |t|^p$ *and* $Q(t) = |t|^q$*. Then the condition* $Q \in \mathcal{A}_M$ *is equivalent to the following conditions :*

1) $2 - \frac{1}{N} < p < N$

2) $q < \overline{q} = \frac{N(p-1)}{N-1}$.

Remark 5.4.5. *In the case where* $M(t) = |t|^p$*. The Corollary 5.4.1 gives the result of Boccardo [45] (i.e.* $u \in W_0^{1,q}(\Omega),\ \ \forall\ \ q < \frac{N(p-1)}{N-1}$*).*

5.4.4 Proof of Theorem 5.2.2

Approximate problem.

Let us defined the following sequence problems

$$\begin{cases} u_n \in K_\psi, \\ \langle Au_n, u_n - v\rangle + \displaystyle\int_\Omega \phi(T_n(u_n))\nabla(u_n - v)\,dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n(u_n - v)\,dx, \\ \forall v \in K_\psi. \end{cases} \tag{5.4.36}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. Applying Lemma 2.2.1, this approximate problem has at last one solution.

A priori estimates.

Let $k > 0$. Taking $u_n - T_k(u_n - v_0)$ as test function in (5.4.36), we obtain for n large enough

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\,dx + \int_\Omega \phi(u_n)\nabla T_k(u_n - v_0)\,dx \leq \int_\Omega f_n T_k(u_n - v_0)\,dx.$$

Since $\nabla T_k(u_n - v_0)$ is identically zero on the set where $|u_n - v_0| > k$, hence we can write

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)\,dx \leq \int_{\{|u_n - v_0|\leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n|\,dx + \int_{\{|u_n - v_0|\leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0|\,dx + \int_\Omega f_n T_k(u_n - v_0)\,dx.$$

Now observe that we have (for $0 < c < 1$),

$$\begin{aligned} \int_{\{|u_n - v_0|\leq k\}} a(x, u_n, \nabla u_n)\nabla u_n\,dx &\leq c\int_{\{|u_n - v_0|\leq k\}} a(x, u_n, \nabla u_n)\frac{\nabla v_0}{c}\,dx \\ &+\int_{\{|u_n - v_0|\leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n|\,dx \\ &+\int_{\{|u_n - v_0|\leq k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0|\,dx + \int_\Omega f_n T_k(u_n - v_0)\,dx. \end{aligned} \tag{5.4.37}$$

By using (A_3), we get

$$\begin{aligned} c\int_{\{|u_n - v_0|\leq k\}} & a(x, u_n, \nabla u_n)\frac{\nabla v_0}{c}\,dx \\ &\leq c\left\{\int_{\{|u_n - v_0|\leq k\}} a(x, u_n, \nabla u_n)\nabla u_n\,dx \right. \\ &\left. - \int_{\{|u_n - v_0|\leq k\}} a(x, u_n, \frac{\nabla v_0}{c})(\nabla u_n - \frac{\nabla v_0}{c})\,dx\right\} \end{aligned} \tag{5.4.38}$$

which yields, thanks (5.4.37),

$$
\begin{aligned}
(1-c)\int_{\{|u_n-v_0|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx &\le \int_{\{|u_n-v_0|\le k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n|\,dx\\
&+\int_{\{|u_n-v_0|\le k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0|\,dx+\int_\Omega f_nT_k(u_n-v_0)\,dx\\
&-c\int_{\{|u_n-v_0|\le k\}} a(x,u_n,\frac{\nabla v_0}{c})(\nabla u_n-\frac{\nabla v_0}{c})\,dx.
\end{aligned}
\tag{5.4.39}
$$

Since $\frac{\nabla v_0}{c}\in(E_M(\Omega))^N$, using (A_2) and the Young's inequality, we have

$$
\begin{aligned}
(1-c)\int_{\{|u_n-v_0|\le k\}} & a(x,u_n,\nabla u_n)\nabla u_n\,dx\\
&\le \tfrac{\alpha(1-c)}{2}\int_{\{|u_n-v_0|\le k\}} M(|\nabla u_n|)\,dx+c_3(k).
\end{aligned}
\tag{5.4.40}
$$

where $c_3(k)$ is a positive constant which depending only in k.
Using also (A_4') we obtain

$$
\frac{\alpha(1-c)}{2}\int_{\{|u_n-v_0|\le k\}} M(|\nabla u_n|)\,dx\le c_3(k).
$$

Moreover, from $\{|u_n|\le k\}\subset\{|u_n-v_0|\le k+\|v_0\|_\infty\}$, we conclude that

$$
\int_\Omega M(|\nabla T_k(u_n)|)\,dx\le c_4(k). \tag{5.4.41}
$$

Almost everywhere convergence of $(u_n)_n$.

Let $k>h\ge\|v_0\|_\infty$. By using $u_n-T_k(u_n-T_h(u_n))$ as test function in (5.4.36), we obtain for n large enough

$$
\begin{aligned}
\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-T_h(u_n))\,dx&+\int_\Omega \phi(u_n)\nabla T_k(u_n-T_h(u_n))\,dx\\
&\le\int_\Omega f_nT_k(u_n-T_h(u_n))\,dx.
\end{aligned}
$$

The second term of the left hand side of the last inequality vanishes for n large enough by virtue of Lemma 1.1.10. This and reasoning as in the proof of Proposition 4.3.3 of Chapter IV, we can easily prove that u_n converges almost everywhere to some function measurable u.

Almost everywhere convergence of the gradient.

We fix $k\ge\|v_0\|_\infty$. By using (A_5) there exists a sequence $v_j\in K_\psi\cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergences in $W_0^1L_M(\Omega)$.
By using the test function $u_n-h_m(u_n)(T_k(u_n)-T_k(v_j))$ in (5.4.36) it is easily adapted from that given in step 5 of proof of Theorem 2.2.1, we prove that

$$
\nabla u_n\to\nabla u\ \text{ a.e. in }\ \Omega.
$$

Finally, we can pass to the limit by applying the same technique used in the passage to the limit in Theorem 2.2.1.
This completes the proof of Theorem 5.2.2.

Part II

QUASI-LINEAR DEGENERATED PROBLEMS

This part is constituted of the following Chapters:

Chapter VI

Preliminaries

Chapter VII

Quasi-linear degenerated equations with L^1 data and without coercivity in perturbation terms

Chapter VIII

Existence of solutions for degenerated unilateral problems in L^1 having lower order terms with natural growth

Introduction and summary of the second part

II.1 Preliminaries

In this Chapter, we recall some preliminary results from weight Sobolev spaces we will need.

II.2 Quasi-linear degenerated equations with L^1 data and without coercivity in perturbation terms

The objective of this Chapter is to study the existence of solution for a strongly non-linear degenerated problem associated to the equation,

$$Au + g(x, u, \nabla u) = f - \mathrm{div}F \qquad (II.1.1)$$

where g is a nonlinearity which satisfies the following growth condition :

$$|g(x, s, \xi)| \leq b(|s|)\left(c(x) + \sum_{i=1}^{N} w_i|\xi_i|^p\right), \qquad (II.1.2)$$

and verifying a sign condition on u a.e.,

$$g(x, s, \xi)s \geq 0. \qquad (II.1.3)$$

but without assuming any coercivity condition.
As regards the second member, we suppose that $f \in L^1(\Omega)$ and $F \in \Pi_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.
The family $w = \{w_i, 0 \leq i \leq N\}$ is a collection of weight functions on Ω.
The variational case ($f \in W^{-1,p'}(\Omega, w^{1-p'})$) with $g = 0$ and $F = 0$ is studied in 1998 by Drabek, Kufner and Nicolosi [68] by using the degree theory in weight Sobolev spaces.
The same variational case, but with $g \neq 0$ is treated in 2003 [18] by Akdim, Azroul and Benkirane by using the approach of sup and sub-solutions.
The same authors have studied in 2002 [17] and 2004 [20] the case of $f \in L^1(\Omega)$ with approaches respectively sup and sub solutions and strong convergence of truncation in weight Sobolev Space under a coercivity condition on g of type

$$|g(x, s, \xi)| \geq \beta \sum_{i=1}^{N} w_i|\xi_i|^p \quad \text{pour} \quad |s| \geq \gamma. \qquad (II.1.4)$$

Moreover, the weight of Hardy σ (that appears in Hardy inequality) satisfies the following integrability condition:

$$\sigma^{1-q'} \in L^1_{\mathrm{loc}}(\Omega), \qquad (II.1.5)$$

with q' is the conjugate exponent of Hardy parameter q.
Our goal in this Chapter is to generalize the work cited above where $F \neq 0$ and without the conditions (II.1.4) et (II.1.5).
The contents of this Chapter is published in the journal Annales Mathématiques Blaises Pascal.

II.3 Existence of solutions for degenerated unilateral problems in L^1 having lower order terms with natural growth

In this Chapter we will consider the degenerate unilateral problem associated with the equation

$$Au + g(x, u, \nabla u) = f \in L^1(\Omega). \quad (II.2.1)$$

The variational unilateral case ($f \in W^{-1,p'}(\Omega, w^{1-p'})$) is studied in [18] under constraint,

$$\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \quad 1 < q < p + p'. \quad (II.2.2)$$

When $f \in L^1(\Omega)$, the strongly nonlinear degenerated problems associated with (II.2.1) is studied in $2002 - 2004$ in [17, 20] by replacing (II.2.2) by

$$\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \quad 1 < q < \infty, \quad (II.2.3)$$

and under the following coercivity condition :

$$|g(x, s, \xi)| \geq \beta \sum_{i=1}^{N} w_i |\xi_i|^p \quad \text{for} \quad |s| \geq \gamma. \quad (II.2.4)$$

The objective of this Chapter is to study the degenerated problem associated to (II.2.1) without assumptions (II.2.2)-(II.2.4). For this we replace the classical coercivity of the operator A by another of type

$$a(x, s, \xi)(\xi - \nabla v_0) \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p - \delta(x), \quad (II.2.5)$$

with $v_0 \in K_\psi \cap L^\infty(\Omega), \delta \in L^1(\Omega)$ and introduced an approximation of the nonlinearity g of the form:

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|} \theta_n(x)$$

with $\theta_n(x) = nT_{1/n}(\sigma^{1/q}(x))$.
The results of this Chapter are the subject of an article published in the journal Portugaliae Mathematica (voir [15]).

Chapter 6

Preliminaries

6.1. Weighted Sobolev spaces

6.1.1. Weighted Lebesgue spaces

Let Ω be a bounded open subset of $I\!R^N (N \geq 2)$. Let $1 < p < \infty$ and let $w = \{w_i(x); i = 0, ..., N\}$, be a vector of weight functions i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that,

$$w_i \in L^1_{loc}(\Omega) \tag{6.1.1}$$

and

$$w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega), \quad \text{for } \ 0 \leq i \leq N. \tag{6.1.2}$$

We define the weighted Lebesgue space with weight γ in Ω as,

$$L^p(\Omega, \gamma) = \{u(x), u\gamma^{\frac{1}{p}} \in L^p(\Omega)\},$$

we endow it

$$\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p \gamma(x)\, dx \right)^{\frac{1}{p}}.$$

Let us give the following Lemmas which are needed below.

Lemma 6.1.1 *[20]. Let $g \in L^r(\Omega, \gamma)$ and let $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{\Omega,\gamma} \leq c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega, \gamma)$.*

Lemma 6.1.2. *Let $g : \Omega \times I\!R^m \to I\!R$ a Carathéodory function. Let $w_0, w_1, ..., w_m$ be weight functions on Ω. Then the corresponding Nemytskii operator G maps continuously $\prod_{i=1}^{m} L^{p_i}(\Omega, w_i)$ into $L^p(\Omega, w_0)$*
if and only if g satisfies:

$$|g(x, s)| \leq a(x) w_0^{-\frac{1}{p}} + c w_0^{-\frac{1}{p}} \sum_{i=1}^{m} |s_i|^{\frac{p_i}{p}} w_i^{\frac{1}{p}}$$

for i.e. $x \in \Omega$ and every $s \in I\!R^m$ with a fixed (nonnegative) function $a \in L^p(\Omega)$ and a fixed nonnegative constant c.

6.1.2. Weighted Sobolev spaces

We denote by $W^{1,p}(\Omega, w)$ the weighted Sobolev space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \text{ for all } i = 1, ..., N.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p w_0 \, dx + \sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx \right)^{\frac{1}{p}}. \qquad (6.1.3)$$

To deal with the Dirichlet problem, we use the space $W_0^{1,p}(\Omega, w)$ defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm (6.1.3). Note that $\mathcal{C}_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(W_0^{1,p}(\Omega, w), \|.\|_{1,p,w})$ is a reflexive Banach space.
We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}, \quad i = 1, ..., N$ and p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$. For more details we refer the reader to [68].
We now introduce the functional spaces we will need in our work:

$$\mathcal{T}_0^{1,p}(\Omega, w) = \Big\{ u : \Omega \to I\!R \text{ measurable}, \quad T_k(u) \in W_0^{1,p}(\Omega, w) \text{ for all } k > 0 \Big\}.$$

We give the following Lemma which is a generalization of Lemma 2.1 [34] in weighted Sobolev spaces (its proof is a slight modification of the original Proof [34]).

Lemma 6.1.3. *For every $u \in \mathcal{T}_0^{1,p}(\Omega, w)$, there exists a uniqne measurable function $v : \Omega \longrightarrow I\!R^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } \Omega, \quad \textit{for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 6.1.4. *Let $\lambda \in I\!R$ and let u and v be two functions which are finite almost everywhere, and which belongs to $\mathcal{T}_0^{1,p}(\Omega, w)$. Then*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \ \textit{a.e. in } \Omega$$

where ∇u, ∇v and $\nabla(u+\lambda v)$ are the gradients of u, v and $u+\lambda v$ introduced in Lemma 6.1.3.

The proof of this Lemma is similar to the proof of Lemma 2.12 [66] in non weighted case.

Definition 6.1.1. *Let Y be a reflexive Banach space. A bounded operator B from Y to its dual Y^* is called pseudo-monotone if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y, and $\limsup_{n\to+\infty} \langle Bu_n, u_n - u\rangle \leq 0$, we have*

$$\liminf_{n\to+\infty} \langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle, \quad \forall\, v \in Y.$$

Now, we state the following assumptions.

(H_1)-The expression,

$$\|u\| = \left(\sum_{i=1}^{N}\int_{\Omega} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}} \tag{6.1.4}$$

is a norm on $W_0^{1,p}(\Omega, w)$ equivalent to the norm (6.1.3).
-There exists a weight function σ strictly positive a.e. in Ω and a parameter q, $1 < q < \infty$, such that the Hardy inequality

$$\left(\int_{\Omega} |u|^q \sigma(x)\, dx\right)^{\frac{1}{q}} \leq C\left(\sum_{i=1}^{N}\int_{\Omega} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}, \tag{6.1.5}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $C > 0$ independent of u. Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma) \tag{6.1.6}$$

determined by the inequality (6.1.5) is compact.
Note that, $(W_0^{1,p}(\Omega, w), \|u\|)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 6.1.1. *Assume that $w_0(x) = 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, \infty[\cap[\frac{1}{p-1}, \infty[$ such that $w_i^{-\nu} \in L^1(\Omega)$ for all $i = 1, ..., N$ (which is stronger than (6.1.2)). Then*

$$\|u\| = \left(\sum_{i=1}^{N}\int_{\Omega} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it is equivalent to (6.1.3). Moreover

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega)$$

for all $1 \leq q < p_1^*$ if $p\nu < N(\nu + 1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu + 1)$, where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see [68]). Thus the hypotheses (H_1) is satisfies for $\sigma \equiv 1$.

Remark 6.1.2. *If we use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \text{dist}(x, \partial\Omega)$ and set*

$$w(x) = d^{\lambda}(x), \quad \sigma(x) = d^{\mu}(x).$$

In this case, the Hardy inequality reads

$$\left(\int_{\Omega} |u|^q d^{\mu}(x)\, dx\right)^{\frac{1}{q}} \leq C\left(\int_{\Omega} |\nabla u|^p d^{\lambda}(x)\, dx\right)^{\frac{1}{p}}$$

(i) For, $1 < p \le q < \infty$,

$$\lambda < p-1, \quad \frac{N}{q}-\frac{N}{p}+1 \ge 0, \quad \frac{\mu}{q}-\frac{\lambda}{p}+\frac{N}{q}-\frac{N}{p}+1>0. \tag{6.1.7}$$

(ii) For, $1 \le q < p < \infty$,

$$\lambda < p-1, \quad \frac{\mu}{q}-\frac{\lambda}{p}+\frac{1}{q}-\frac{1}{p}+1>0. \tag{6.1.8}$$

The conditions (6.1.7) or (6.1.8) are sufficient for the compact imbedding (6.1.6) to hold (see for example ([67], Example 1), ([68], Example 1.5, p.34), and ([88], Theorems 19.17 and 19.22)).

Now, we give the following technical Lemmas which are needed later.

Lemma 6.1.5 *[20]. Assume that* (H_1) *holds. Let* $F : I\!R \to I\!R$ *be uniformly Lipschitzian, with* $F(0)=0$. *Let* $u \in W_0^{1,p}(\Omega, w)$. *Then* $F(u) \in W_0^{1,p}(\Omega, w)$. *Moreover, if the set* D *of discontinuity points of* F' *is finite, then*

$$\frac{\partial (F\circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \quad in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

From the previous Lemma, we deduce the following.

Lemma 6.1.6 *[20]. Assume that* (H_1) *holds. Let* $u \in W_0^{1,p}(\Omega, w)$, *and let* $T_k(u)$ *be the usual truncation (* $k \in I\!R^+$ *) then* $T_k(u) \in W_0^{1,p}(\Omega, w)$. *Moreover, we have*

$$T_k(u) \to u \quad strongly\ in \quad W_0^{1,p}(\Omega, w).$$

6.2. Notations

In the sequel, we use the following notations
Denoting by $\epsilon(n,j,h)$ any quantity such that

$$\lim_{h\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,h)=0.$$

If the quantity we consider does not depend on one parameter among n, j and h, we will omit the dependence on the corresponding parameter: as an example, $\epsilon(n,h)$ is any quantity such that

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\epsilon(n,h)=0.$$

Finally, we will denote (for example) by $\epsilon_h(n,j)$ a quantity that depends on n, j, h and is such that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon_h(n,j)=0$$

for any fixed value of h.

Chapter 7

Quasi-linear degenerated equations with L^1 datum and without coercivity in perturbation terms[1]

In this Chapter, we shall prove the existence of solutions for some quasi-linear degenerated elliptic equations of the type

$$Au + g(x, u, \nabla u) = f - \text{div}F$$

with natural growth condition on the coefficient and without coercivity condition on the nonlinear term g. The terms of second member belong respectively to $L^1(\Omega)$, and $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.

7.1. Introduction

Let Ω be a bounded subset of $I\!R^N$, $1 < p < \infty$, and w be a collection of weight functions on Ω : $w = \{w_i(x);\ i = 0, ..., N\}$, i.e., each w_i is a measurable and positive function everywhere on Ω and satisfying some integrability conditions (see Section 3). Let us consider the second order differential operator,

$$Au = -\text{div}(a(x, u, \nabla u)) \tag{7.1.1}$$

In this setting Drabek, Kufner and Mustonen in [67] have proved that the Dirichlet problem associated with the equation,

$$Au = h \in W^{-1,p'}(\Omega, w^*)$$

has at least one solution u in $W_0^{1,p}(\Omega, w)$ (see also [19], where uniquely the large monotonicity is used).
Now, consider the following Dirichlet problems associated to the equations,

$$Au + g(x, u, \nabla u) = f - \text{div}F \ \text{ in } \ \Omega. \tag{7.1.2}$$

[1]Annales Mathématiques Blaises Pascal 11, 47-66 (2004)

In the variational case (i.e., the second member belongs to $\in W^{-1,p'}(\Omega, w^*)$), an existence Theorem has recently proved in [18], where the authors have used the approach based on the strong convergence of the positive part u_ε^+ (resp. negative part u_ε^-).
In the case where $f \in L^1(\Omega), F = 0$, they also give an existence result in [20] if the nonlinearity g satisfies further the following coercivity condition,

$$|g(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i |\xi_i|^p \quad \text{for} \quad |s| \geq \gamma. \tag{7.1.3}$$

the result is proved by using another approach based on the strong convergence of truncation.
It is our purpose, in this Chapter, to prove an existence result for some class of problem of the kind (7.1.2), without assuming the coercivity condition (7.1.3). Moreover, we didn't suppose any integrability condition for function σ. For different approach used in the setting of Orlicz Sobolev space the reader can referred to [43], and for same results in the L^p case, to [102].

7.2. Main results

Let A be the nonlinear operator from $W_0^{1,p}(\Omega, w)$ into the dual $W^{-1,p'}(\Omega, w^*)$ defined as

$$Au = -\text{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfying the following assumptions:
(H_2) For $i = 1, ..., N$

$$|a_i(x,s,\xi)| \leq w_i^{\frac{1}{p}}(x)[k(x) + \sigma^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}] \tag{7.2.1}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \quad \text{for all} \quad \xi \neq \eta \in I\!R^N, \tag{7.2.2}$$

$$a(x,s,\xi)\xi \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p. \tag{7.2.3}$$

Where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, a positive constant.
(H_3) $g(x,s,\xi)$ is a Carathéodory function satisfying

$$g(x,s,\xi).s \geq 0 \tag{7.2.4}$$

$$|g(x,s,\xi)| \leq b(|s|) \left(c(x) + \sum_{i=1}^{N} w_i(x)|\xi_i|^p \right), \tag{7.2.5}$$

where $b : I\!R^+ \to I\!R^+$ is a positive increasing function and $c(x)$ is a positive function which belong to $L^1(\Omega)$.
For the nonlinear Dirichlet boundary value problem (7.1.2), we state our main result as follows.

Theorem 7.2.1. *Assume that* $(H_1)-(H_3)$ *holds and let* $f \in L^1(\Omega), F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$. *Then there exists at least one solution of* (7.1.2) *in the following sense:*

$$(P)\begin{cases} T_k(u) \in W_0^{1,p}(\Omega, w), \qquad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad\qquad \displaystyle\leq \int_\Omega fT_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx \\ \forall v \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}$$

Remark 7.2.1. *The statement of Theorem* 2.1 *generalizes in weighted case the analogous one in [102].*

7.3. Proof of main results

The following Lemma play an important rôle in the proof of our main result,

Lemma 7.3.1 *[17]. Assume that* (H_1) *and* (H_2) *are satisfied, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p}(\Omega, w)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega, w)$ *and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0$$

then, $u_n \to u$ *in* $W_0^{1,p}(\Omega, w)$.

To prove the existence Theorem we proceed by Sections.

7.3.1. Approximate problem

Let f_n a regular function such that f_n strongly converges to f in $L^1(\Omega)$.
We consider the sequence of approximate problems:

$$\begin{cases} u_n \in W_0^{1,p}(\Omega, w), \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla v\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)v\,dx \\ \qquad\qquad\qquad \displaystyle= \int_\Omega f_n v\,dx + \int_\Omega F\nabla v\,dx \\ \forall v \in W_0^{1,p}(\Omega, w). \end{cases} \tag{7.3.1}$$

where $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}\theta_n(x)$ with $\theta_n(x) = nT_{1/n}(\sigma^{1/q}(x))$.
Note that $g_n(x,s,\xi)$ satisfies the following conditions

$$g_n(x,s,\xi)s \geq 0, \quad |g_n(x,s,\xi)| \leq |g(x,s,\xi)| \ \text{ and } \ |g_n(x,s,\xi)| \leq n.$$

We define the operator $G_n\ : W_0^{1,p}(\Omega, w) \longrightarrow W^{-1,p'}(\Omega, w^*)$ by,

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)v\,dx$$

and

$$\langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\,dx$$

Thanks to Hölder's inequality, we have for all $u \in X$ and $v \in X$,

$$\begin{aligned}\left|\int_\Omega g_n(x,u,\nabla u)v\;dx\right| &\leq \left(\int_\Omega |g_n(x,u,\nabla u)|^{q'}\sigma^{-\frac{q'}{q}}\;dx\right)^{\frac{1}{q'}}\left(\int_\Omega |v|^q\sigma\;dx\right)^{\frac{1}{q}}\\ &\leq n^2\left(\int_\Omega \sigma^{q'/q}\sigma^{-q'/q}\;dx\right)^{\frac{1}{q'}}\|v\|_{q,\sigma}\\ &\leq C_n\|v\|_X,\end{aligned} \tag{7.3.2}$$

Lemma 7.3.2. *The operator $B_n = A+G_n$ from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ is pseudo-monotone. Moreover, B_n is coercive, in the following sense:*

$$\frac{< B_n v, v >}{\|v\|} \longrightarrow +\infty \quad if \quad \|v\| \longrightarrow +\infty, v \in W_0^{1,p}(\Omega,w).$$

This Lemma will be proved below.
In view of Lemma 7.3.2, Problem (7.3.1) has a solution by the classical result (cf. Theorem 2.1 and Remark 2.1 in Chapter 2 of [92]).
Taking $v = T_k(u_n)$ as test function in (7.3.1), we have

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx\\ = \int_\Omega f_nT_k(u_n)\,dx + \int_\Omega F\nabla T_k(u_n)\,dx\end{aligned}$$

and by using in fact that $g_n(x,u_n,\nabla u_n)T_k(u_n) \geq 0$, we obtain

$$\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \leq ck + \int_\Omega F\nabla T_k(u_n)\,dx.$$

Thank's to Young's inequality and (7.2.3), one easily has

$$\frac{\alpha}{2}\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p w_i(x)\;dx \leq c_1k. \tag{7.3.3}$$

Now, we prove that u_n converges to some function u locally in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). To prove this, we chow that u_n is a Cauchy sequence in measure in any ball B_R.
Let $k > 0$ large enough, we have

$$\begin{aligned}k meas(\{|u_n| > k\}\cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx \leq \int_{B_R}|T_k(u_n)|\,dx\\ &\leq \left(\int_\Omega |T_k(u_n)|^p w_0\,dx\right)^{\frac{1}{p}}\left(\int_{B_R} w_0^{1-p'}\,dx\right)^{\frac{1}{q'}}\\ &\leq c_2\left(\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p w_i(x)\;dx\right)^{\frac{1}{p}}\\ &\leq c_1k^{\frac{1}{p}}.\end{aligned}$$

Which implies that

$$meas(\{|u_n| > k\} \cap B_R) \leq \frac{c_1}{k^{1-\frac{1}{p}}} \quad \forall k > 1. \tag{7.3.4}$$

We have, for every $\delta > 0$,

$$\begin{aligned} meas(\{|u_n - u_m| > \delta\} \cap B_R) \quad &\leq \quad meas(\{|u_n| > k\} \cap B_R) + meas(\{|u_m| > k\} \cap B_R) \\ &\qquad + meas\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned} \tag{7.3.5}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, w)$, there exists some $v_k \in W_0^{1,p}(\Omega, w)$, such that

$$\begin{array}{lll} T_k(u_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,p}(\Omega, w) \\ T_k(u_n) \to v_k & \text{strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega. \end{array}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then, by (7.3.4) and (7.3.5), there exists some $k(\varepsilon) > 0$ such that

$$meas(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon \text{ for all } n, m \geq n_0(k(\varepsilon), \delta, R).$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{array}{lll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega, w), \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega. \end{array}$$

Which yields, by using (7.2.1), for all $k > 0$ there exists a function $h_k \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i)$, such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i). \tag{7.3.6}$$

7.3.2. Strong convergence of truncations

We fix $k > 0$, and let

$$\gamma = (\frac{b(k)}{\alpha})^2, \ \varphi_k(s) = se^{\gamma s^2}.$$

It is well known that

$$\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2}, \quad \forall s \in I\!R. \tag{7.3.7}$$

Here, we define the function $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ where $h > k > 0$. We define the following function as

$$v_n = \varphi_k(w_n). \tag{7.3.8}$$

The use of v_n as test function in (7.3.1), gives

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \nabla \varphi_k(w_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) \varphi_k(w_n)\, dx \\ = \int_\Omega f_n \varphi_k(w_n)\, dx + \int_\Omega F \nabla \varphi_k(w_n)\, dx. \end{aligned}$$

It follows that

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla w_n\varphi_k'(w_n)\,dx &+ \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_n)\,dx\\ &= \int_\Omega f_n\varphi_k(w_n)\,dx + \int_\Omega F\nabla\varphi_k(w_n)\,dx.\end{aligned} \tag{7.3.9}$$

Note that, $\nabla w_n = 0$ on the set where $|u_n| > h+4k$, therefore, setting $m = 4k+h$, and denoting by $\varepsilon_h^1(n), \varepsilon_h^2(n), ...$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h, we get, by (7.3.9),

$$\begin{aligned}\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_n\varphi_k'(w_n)\,dx &+ \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_n)\,dx\\ &= \int_\Omega f_n\varphi_k(w_n)\,dx + \int_\Omega F\nabla\varphi_k(w_n)\,dx.\end{aligned}$$

Since $\varphi_k(w_n)g_n(x,u_n,\nabla u_n) > 0$ on the subset $\{x\in\Omega, |u_n(x)| > k\}$, we deduce that

$$\begin{aligned}\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_n\varphi_k'(w_n)\,dx &+ \int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_n)\,dx\\ &\le \int_\Omega f_n\varphi_k(w_n)\,dx + \int_\Omega F\nabla\varphi_k(w_n)\,dx.\end{aligned} \tag{7.3.10}$$

Splitting the first integral on the left hand side of (7.3.10) where $|u_n| \le k$ and $|u_n| > k$, and by using (7.2.3), we can write,

$$\begin{aligned}\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))&\nabla w_n\varphi_k'(w_n)\,dx\\ &\ge \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_n)\,dx\\ &\quad -C_3(k)\int_{\{|u_n|>k\}} |a(x,T_m(u_n),\nabla T_m(u_n))||\nabla T_k(u)|\,dx,\end{aligned} \tag{7.3.11}$$

where $C_3(k) = \varphi_k'(2k)$. Since, for all $i = 1,...,N$, $\frac{\partial(T_k(u))}{\partial x_i}\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $L^p(\Omega,w_i)$ as n tends to infinity while, $(a_i(x,T_m(u_n),\nabla T_m(u_n)))_n$ is bounded in $L^{p'}(\Omega,w_i^{1-p'})$, hence the last term in the previous inequality tends to zero for every h fixed.

Now, observe that

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),&\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_n)\,dx\\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_n)\,dx\\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)\varphi_k'(T_k(u_n)-T_k(u))\,dx\\ &\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)\varphi_k'(w_n)\,dx.\end{aligned} \tag{7.3.12}$$

By the continuity of the Nymetskii operator, we have for all $i = 1,...,N$,

$$a_i(x,T_k(u_n),\nabla T_k(u))\varphi_k'(T_k(u_n)-T_k(u)) \to a_i(x,T_k(u),\nabla T_k(u))\varphi_k'(0)$$

and

$$a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$$

strongly in $L^{p'}(\Omega, w_i^{1-p'})$, while $\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega, w_i)$, and $\frac{\partial(T_k(u))}{\partial x_i}\varphi_k'(w_n)$ $\to \frac{\partial(T_k(u))}{\partial x_i}\varphi_k'(0)$ strongly in $L^p(\Omega, w_i)$.
The second and the third term of the right hand side of (7.3.12) tends respectively to $\int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\varphi_k'(0)\, dx$ and $-\int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\varphi_k'(0)\, dx$.
So that (7.3.11) and (7.3.12) yields

$$\begin{aligned} \int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_n)\, dx \\ \geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ \times[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_n)\, dx + \varepsilon_h^2(n). \end{aligned} \tag{7.3.13}$$

For the second term of the left hand side of (7.3.10), we can estimate as follows

$$\begin{aligned} \left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(w_n)\, dx\right| &\leq \int_{\{|u_n|\leq k\}} b(k)\left(c(x) + \sum_{i=1}^N w_i|\frac{\partial T_k(u_n)}{\partial x_i}|^p\right)|\varphi_k(w_n)|\, dx \\ &\leq b(k)\int_\Omega c(x)|\varphi_k(w_n)|\, dx \\ &+\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_n)|\, dx, \end{aligned} \tag{7.3.14}$$

remark that, we have

$$\begin{aligned} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_n)|\, dx \\ = \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ \times[\nabla T_k(u_n) - \nabla T_k(u)]|\varphi_k(w_n)|\, dx \\ +\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u)|\varphi_k(w_n)|\, dx \\ +\int_\Omega a(x, T_k(u_n), \nabla T_k(u))[\nabla T_k(u_n) - \nabla T_k(u)]|\varphi_k(w_n)|\, dx. \end{aligned} \tag{7.3.15}$$

By Lebesgue's Theorem, we deduce that

$$\nabla T_k(u)|\varphi_k(w_n)| \to \nabla T_k(u)|\varphi_k(T_{2k}(u - T_h(u)))| \text{ strongly in } \prod_{i=1}^N L^p(\Omega, w_i).$$

Which and using (7.3.6) implies that the second term of the right hand side of (7.3.15) tends to

$$\int_\Omega h_k \nabla T_k(u)|\varphi_k(T_{2k}(u - T_h(u)))|\, dx = 0.$$

As in the (7.3.12), the third term of the right hand side of (7.3.15) tends to 0. From (7.3.14) and (7.3.15), we obtain

$$\begin{aligned}&\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_n)\,dx\right|\\ &\qquad\le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\quad \times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_n)|\,dx\\ &\qquad\quad +\varepsilon_h^3(n).\end{aligned} \tag{7.3.16}$$

Now, by the strong convergence of $f_n, F \in (L^{p'}(\Omega, w^{1-p'}))^N$ and in fact that

$$\varphi_k(w_n) \rightharpoonup \varphi_k(T_{2k}(u-T_h(u))) \text{ weakly in } W_0^{1,p}(\Omega,w) \text{ and weakly-}* \text{ in } L^\infty(\Omega). \tag{7.3.17}$$

Combining (7.3.10), (7.3.13), (7.3.16) and (7.3.17), we get

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad \times[\nabla T_k(u_n)-\nabla T_k(u)](\varphi_k'(w_n) - \tfrac{b(k)}{\alpha}|\varphi_k(w_n)|)\,dx\\ &\qquad\quad \le b(k)\int_\Omega c(x)\varphi_k(T_{2k}(u-T_h(u)))\,dx + \int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))\,dx,\\ &\qquad\quad + \int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi_k'(T_{2k}(u-T_h(u)))\,dx + \varepsilon_h^5(n).\end{aligned}$$

which and (7.3.7) implies that

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad \times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\ &\qquad \le 2b(k)\int_\Omega c(x)\varphi_k(T_{2k}(u-T_h(u)))\,dx + 2\int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))\,dx,\\ &\qquad\quad +2\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi_k'(T_{2k}(u-T_h(u)))\,dx + \varepsilon_h^5(n).\end{aligned}$$

Hence, passing to the limit over n, we get

$$\begin{aligned}&\limsup_{n\to\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\ &\qquad \le 2b(k)\int_\Omega c(x)\varphi_k(T_{2k}(u-T_h(u)))\,dx + 2\int_\Omega f\varphi_k(T_{2k}(u-T_h(u)))\,dx,\\ &\qquad\quad +2\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi_k'(T_{2k}(u-T_h(u)))\,dx.\end{aligned} \tag{7.3.18}$$

It remains to show, for our purposes, that the all term on the right hand side of (7.3.18) converge to zero as h goes to infinity. The only difficulty that exists is in the last term. For the other terms it suffices to apply Lebesgue's Theorem.
We deal with this term. Let us observe that, if we take $\varphi_k(T_{2k}(u_n - T_h(u_n)))$ as test

function in (7.3.1) and using (7.2.3), we obtain

$$\alpha\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\varphi_k'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\leq\int_{\{h\leq|u_n|\leq 2k+h\}}F\nabla u_n\varphi_k'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\int_\Omega f_n\varphi_k(T_{2k}(u_n-T_h(u_n)))\,dx.$$

Since $g_n(x,u_n,\nabla u_n)\varphi_k(T_{2k}(u_n-T_h(u_n)))\geq 0$, We get

$$\alpha\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\varphi_k'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\leq\int_{\{h\leq|u_n|\leq 2k+h\}}F\nabla u_n\varphi_k'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega f_n\varphi_k(T_{2k}(u_n-T_h(u_n)))\,dx,$$

which yields, thanks to Young's inequalities

$$\tfrac{\alpha}{2}\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\varphi_k'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\leq\int_\Omega f_n\varphi_k(T_{2k}(u_n-T_h(u_n)))\,dx+c_4(k)\int_{\{h\leq|u_n|\}}|w^{\frac{-1}{p}}F|^{p'}\,dx. \tag{7.3.19}$$

On the other hand, by the continuity of φ_k', we have

$$\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\varphi_k'(T_{2k}(u-T_h(u)))\,dx$$
$$\leq c_5(k)\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\,dx.$$

and since the norm is lower semi-continuity and $\varphi_k'\geq 1$, we get

$$\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\varphi_k'(T_{2k}(u-T_h(u)))\,dx$$
$$\leq c_5(k)\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\,dx$$
$$\leq c_5(k)\liminf_{n\to\infty}\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u_n-T_h(u_n))}{\partial x_i}\right|^p w_i\,dx$$
$$\leq c_5(k)\liminf_{n\to\infty}\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_{2k}(u_n-T_h(u_n))}{\partial x_i}\right|^p w_i\varphi_k'(T_{2k}(u_n-T_h(u_n)))\,dx \tag{7.3.20}$$

Combining (7.3.19) and (7.3.20), we deduce

$$\int_\Omega \sum_{i=1}^{N} \left|\frac{\partial T_{2k}(u-T_h(u))}{\partial x_i}\right|^p w_i\varphi_k'(T_{2k}(u-T_h(u)))\,dx$$
$$\leq c_7(k)\liminf_{n\to\infty}\int_{\{h\leq|u_n|\}} |w^{-1/p}F|^{p'}\,dx$$
$$+c_7(k)\liminf_{n\to\infty}\int_\Omega f_n\varphi_k(T_{2k}(u_n-T_h(u_n)))\,dx,$$

consequently, the strong convergence in $L^1(\Omega)$ of f_n and since $|w^{\frac{-1}{p}}F|^{p'} \in L^1(\Omega)$, we have, as first n and then h tend to infinity,

$$\limsup_{h\to\infty}\int_{\{h\leq|u|\leq 2k+h\}} \sum_{i=1}^{N}|\frac{\partial u}{\partial x_i}|^p w_i\varphi_k'(T_{2k}(u-T_h(u)))\,dx = 0,$$

so that

$$\lim_{h\to\infty}\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi_k'(T_{2k}(u-T_h(u)))\,dx = 0.$$

Therefore by (7.3.18), letting h go to infinity, we conclude,

$$\lim_{n\to\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx = 0,$$

which and using Lemma 6.4.1 implies that

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega,w) \ \ \forall k>0. \tag{7.3.21}$$

7.3.3. Passing to the limit

By using $T_k(u_n-v)$ as test function in (7.3.1), with $v\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$, we get

$$\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx$$
$$= \int_\Omega f_nT_k(u_n-v)\,dx + \int_\Omega F\nabla T_k(u_n-v)\,dx. \tag{7.3.22}$$

By Fatou's Lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $\prod_{i=1}^{N} L^{p'}(\Omega,w_i^{1-p'})$ one easily sees that

$$\int_\Omega a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla T_k(u-v)\,dx$$
$$\leq \liminf_{n\to\infty}\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx. \tag{7.3.23}$$

On the other hand, since $F \in (L^{p'}(\Omega, w^{1-p'}))^N$ and the fact that

$$\nabla T_k(u_n - v) \rightharpoonup \nabla T_k(u - v) \text{ weakly in } \prod_{i=1}^{N} L^p(\Omega, w_i),$$

we deduce that

$$\int_\Omega F\nabla T_k(u_n - v)\, dx \longrightarrow \int_\Omega F\nabla T_k(u - v)\, dx \text{ as } n \to \infty.$$

Now, we need to prove that

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega),$$

in particular it is enough to prove the equiintegrable of $g_n(x, u_n, \nabla u_n)$. To this purpose. We take $T_{l+1}(u_n) - T_l(u_n)$ as test function in (7.3.1), we obtain

$$\int_{\{|u_n|>l+1\}} |g_n(x, u_n, \nabla u_n)|\, dx \le \int_{\{|u_n|>l\}} |f_n|\, dx.$$

Let $\varepsilon > 0$. Then there exists $l(\varepsilon) \ge 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}} |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon/2. \tag{7.3.24}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned} \int_E |g_n(x, u_n, \nabla u_n)|\, dx \le & \int_E b(l(\varepsilon)) \left(c(x) + \sum_{i=1}^{N} w_i \left|\frac{\partial(T_{l(\varepsilon)}(u_n))}{\partial x_i}\right|^p \right) dx \\ & + \int_{\{|u_n|>l(\varepsilon)\}} |g_n(x, u_n, \nabla u_n)|\, dx. \end{aligned}$$

In view by (7.3.21) there exists $\eta(\varepsilon) > 0$ such that

$$\begin{aligned} \int_E b(l(\varepsilon)) \left(c(x) + \sum_{i=1}^{N} w_i \left|\tfrac{\partial(T_{l(\varepsilon)}(u_n))}{\partial x_i}\right|^p \right) dx < \varepsilon/2 \\ \text{for all E such that meas } E < \eta(\varepsilon). \end{aligned} \tag{7.3.25}$$

Finally, by combining (7.3.24) and (7.3.25) one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon \text{ for all } E \text{ such that meas } E < \eta(\varepsilon),$$

which allows us, by using (7.3.23), to pass to the limit in (7.3.22).
This completes the proof of Theorem 6.1.

Remark 7.3.1. *Note that, we obtain the existence result without assuming the coercivity condition. However one can overcome this difficulty by introduced the function* $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ *in the test function* (7.3.8).

Proof of Lemma 7.3.2

From Hölder's inequality, the growth condition (7.2.1) we can show that A is bounded, and by using (7.3.2), we have B_n bounded. The coercivity follows from (7.2.3) and (7.2.4). it remain to show that B_n is pseudo-monotone.
Let a sequence $(u_k)_k \in W_0^{1,p}(\Omega, w)$ such that

$$u_k \rightharpoonup u \ \text{ weakly in } \ W_0^{1,p}(\Omega, w),$$

$$B_n u_k \rightharpoonup \chi \ \text{ weakly in } \ W^{-1,p'}(\Omega, w^*),$$

and $\limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi, u\rangle$.
We will prove that

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \ \text{ as } \ k \to +\infty.$$

Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p}(\Omega, w)$, we deduce that $(a(x, u_k, \nabla u_k))_k$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$, then there exists a function $h \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup h \ \text{ weakly in } \ \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}),$$

similarly, it is easy to see that $(g_n(x, u_k, \nabla u_k))_k$ is bounded in $L^{q'}(\Omega, \sigma^{1-q'})$, then there exists a function $k_n \in L^{q'}(\Omega, \sigma^{1-q'})$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup k_n \ \text{ weakly in } \ Ł^{q'}(\Omega, \sigma^{1-q'}).$$

It is clear that, for all $v \in W_0^{1,p}(\Omega, w)$

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to+\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to+\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla v \, dx \\ &\quad + \lim_{k\to+\infty} \int_\Omega g_n(x, u_k, \nabla u_k).v \, dx. \end{aligned}$$

Consequently, we get

$$\langle \chi, v\rangle = \int_\Omega h\nabla v \, dx + \int_\Omega k_n.v \, dx, \quad \forall \ v \in W_0^{1,p}(\Omega, w). \tag{7.3.26}$$

On the other hand, we have

$$\int_\Omega g_n(x, u_k, \nabla u_k).u_k \, dx \longrightarrow \int_\Omega k_n.u \, dx, \tag{7.3.27}$$

and, by hypotheses, we have

$$\begin{aligned} \limsup_{k\to\infty} &\left\{ \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k \, dx + \int_\Omega g_n(x, u_k, \nabla u_k).u_k \, dx \right\} \\ &\qquad \le \int_\Omega h\nabla u \, dx + \int_\Omega k_n.u \, dx, \end{aligned}$$

therefore

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \le \int_\Omega h\nabla u\,dx. \tag{7.3.28}$$

So that it is enough to prove that

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \ge \int_\Omega h\nabla u\,dx.$$

By condition (7.2.2), we have

$$\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,dx \ge 0.$$

Consequently

$$\begin{aligned}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \;&\ge\; -\int_\Omega a(x,u_k,\nabla u)\nabla u\,dx+\int_\Omega a(x,u_k,\nabla u_k)\nabla u\,dx\\ &\quad +\int_\Omega a(x,u_k,\nabla u)\nabla u_k\,dx,\end{aligned}$$

hence

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \ge \int_\Omega h\nabla u\,dx.$$

This implies by using (7.3.28)

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx = \int_\Omega h\nabla u\,dx. \tag{7.3.29}$$

By means of (7.3.26), (7.3.27) and (7.3.29), we obtain

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as} \quad k \longrightarrow +\infty.$$

Corollary 7.3.1. *Let $1<p<\infty$. Assume that the hypothesis $(H_1)-(H_3)$ holds, let f_n any sequence of function in $L^1(\Omega)$ converge to f weakly in $L^1(\Omega)$ and let u_n the solution of the following problem*

$$(P'_n)\begin{cases} T_k(u_n)\in W_0^{1,p}(\Omega,w), \quad g(x,u_n,\nabla u_n)\in L^1(\Omega)\\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\\ \qquad\qquad\qquad \displaystyle\le \int_\Omega f_n T_k(u_n-v)\,dx,\\ \forall\ \ v\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega),\ \forall k>0.\end{cases}$$

Then there exists a subsequence of u_n still denoted u_n such that u_n converges to u almost everywhere and $T_k(u_n)\rightharpoonup T_k(u)$ strongly in $W_0^{1,p}(\Omega,w)$, further u is a solution of the problem (P) (with $F=0$).

Proof of Corollary

We give the proof briefly.

STEP 1. A priori estimates.

We proceed as previous, we take $v = 0$ as test function in (P'_n), we get

$$\int_\Omega \sum_{i=1}^{N} w_i \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p \, dx \leq C_8 k. \tag{7.3.30}$$

Hence, by the same method used in the first step in the proof of Theorem 2.1 there exists a function u (with $T_k(u) \in W_0^{1,p}(\Omega, w)$ $\forall k > 0$) and a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup T_k(u) \ \text{ weakly in } \ W_0^{1,p}(\Omega, w), \ \forall k > 0.$$

STEP 2. Strong convergence of truncation

The choice of $v = T_s(u_n - \phi(w_n))$ as test function in (P'_n), we get, for all $l > 0$

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_l(u_n - T_s(u_n - \phi(w_n))) \, dx + \int_\Omega g(x, u_n, \nabla u_n) T_l(u_n - T_s(u_n - \phi(w_n)) \, dx$$
$$\leq \int_\Omega f_n T_l(u_n - T_s(u_n - \phi(w_n))) \, dx.$$

Which implies that

$$\int_{\{|u_n - \phi(w_n)| \leq s\}} a(x, u_n, \nabla u_n)\nabla T_l(\phi(w_n))$$
$$+ \int_\Omega g(x, u_n, \nabla u_n) T_l(u_n - T_s(u_n - \phi(w_n))) \, dx$$
$$\leq \int_\Omega f_n T_l(u_n - T_s(u_n - \phi(w_n))) \, dx.$$

Letting s tends to infinity and choosing l large enough, we deduce

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla \phi(w_n) \, dx + \int_\Omega g(x, u_n, \nabla u_n)\phi(w_n) \, dx$$
$$\leq \int_\Omega f_n \phi(w_n) \, dx,$$

the rest of the proof of this step is the same as in step 2 of the proof of Theorem 3.1.

STEP 3. Passing to the limit

This step is similarly to the step 3 of the proof of Theorem 3.1, by using the Egorov's Theorem in the last term of (P'_n).

Remark 7.3.2. *In the case where $F = 0$, if we suppose that the second member are nonnegative, then we obtain a nonnegative solution.*

Proof. If we take $v = T_h(u^+)$ in (P), we have

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_h(u^+))\,dx \\ + \int_\Omega g(x,u,\nabla u)T_k(u - T_h(u^+))\,dx \\ \leq \int_\Omega fT_k(u - T_h(u^+))\,dx.$$

Since $g(x,u,\nabla u)T_k(u - T_h(u^+)) \geq 0$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_h(u^+))\,dx \leq \int_\Omega fT_k(u - T_h(u^+))\,dx,$$

we remark also, by using $f \geq 0$

$$\int_\Omega fT_k(u - T_h(u^+))\,dx \leq \int_{\{u \geq h\}} fT_k(u - T_h(u))\,dx.$$

On the other hand, thanks to (7.2.3), we conclude

$$\alpha \int_\Omega \sum_{i=1}^N w_i |\frac{\partial T_k(u^-)}{\partial x_i}|^p\,dx \leq \int_{\{u \geq h\}} fT_k(u - T_h(u))\,dx.$$

Letting h tend to infinity, we can easily deduce

$$T_k(u^-) = 0, \quad \forall\, k > 0,$$

which implies that

$$u \geq 0.$$

Chapter 8

Existence of solutions for degenerated unilateral problems in L^1 having lower order terms with natural growth[1]

This Chapter deals with the quasi-linear degenerated elliptic unilateral problem associated to the following equations,

$$Au + g(x, u, \nabla u) = f,$$

where A is a Leray-Lions operator acting from the weighted Sobolev space $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and now growth with respect to s but it satisfies a sign condition on s, i.e. $g(x, s, \xi).s \geq 0$ for every $s \in I\!R$. The datum f is assumed in $L^1(\Omega)$.

8.1. Introduction

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$. Let $f \in L^1(\Omega)$. Consider the following nonlinear Dirichlet problem:

$$Au + g(x, u, \nabla u) = f \tag{8.1.1}$$

where $Au = -\text{div}(a(x, u, \nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega, w)$ into its dual and $g(x, u, \nabla u)$ is a nonlinearity satisfying the following natural growth,

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + \sum_{i=1}^{N} w_i |\xi_i|^p)$$

and the sign condition,

$$g(x, s, \xi).s \geq 0.$$

[1]Portugaliae Mathematica vol. 65, Fasc. $1(2008), 95 - 120$.

In the particular case where $g(x, u, \nabla u) = -C_0|u|^{p-2}u$, the following degenerated equation,

$$-\text{div}(a(x, u, \nabla u)) - C_0|u|^{p-2}u = f(x, u, \nabla u)$$

has been studied by Drabek-Nicolosi [70] under more degeneracy and some additional assumptions on f and $a(x, s, \xi)$.

Concerning the degenerated unilateral problem associated to the equation (8.1.1), an existence result is proved in [18], with the second member f lying in the dual space $W^{-1,p'}(\Omega, w^*)$. To do this, the authors have introduced the following integrability condition

$$\sigma^{1-q'} \in L^1_{loc}(\Omega) \text{ with } 1 < q < p + p', \tag{8.1.2}$$

where σ is some function and q is some parameter which appears in the Hardy type inequality (see (H_1) in Chapter VI). Recently, in [20] the authors have studied the existence solution for the degenerated problem associated to the equation (8.1.1), where the right hand side f is assumed to belong to $W^{-1,p'}(\Omega, w^*)$ (resp. to $L^1(\Omega)$) and where the integrability condition (8.1.2) is replaced by the weaker condition

$$\sigma^{1-q'} \in L^1_{loc}(\Omega) \text{ with } 1 < q < +\infty. \tag{8.1.3}$$

Note that, in this last case (where $f \in L^1(\Omega)$), the authors have also assumed that $g(x, s, \xi)$ has an "exact natural growth", i.e.,

$$|g(x, s, \xi)| \geq \gamma \sum_{i=1}^{N} w_i |\xi_i|^p \text{ for } |s| \text{ sufficiently large.} \tag{8.1.4}$$

Now, let $K_\psi = \{v \in W_0^{1,p}(\Omega, w);\ v \geq \psi \ \ a.e. \text{ in } \Omega\}$, with ψ a measurable function on Ω, we consider the following problem

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega, w), u \geq \psi \ a.e. \text{ in } \Omega, g(x, u, \nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega g(x, u, \nabla u)T_k(u - v)\, dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f T_k(u - v)\, dx, \\ \forall \ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0, \end{cases} \tag{8.1.5}$$

where T_k is the truncation operator at height $k > 0$ (see (1.1.6)).

The aim of this Chapter is to study the existence solution of the problem (8.1.5) without using the condition (8.1.2), (8.1.3) and (8.1.4).

Note that, the hypothesis (8.1.2) used in [18] have plied an important role to assure the boundedness, coercivity and pseudo-monotonicity of the corresponding operators in the approximate problem and also to prove the boundedness of the approximate solution u_n (see [18]).

To overcome this difficulty, in the present Chapter, we change the classical coercivity condition, i.e.,

$$a(x, s, \xi)\xi \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p$$

by the following one

$$a(x,s,\zeta)(\zeta-\nabla v_0)\geq \alpha\sum_{i=1}^{N} w_i(x)|\zeta_i|^p-\delta(x). \tag{8.1.6}$$

Also, we approximate the nonlinearity g by

$$g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}\theta_n(x) \tag{8.1.7}$$

where $\theta_n(x)=nT_{1/n}(\sigma^{1/q}(x))$.
Furthermore, we eliminate the condition (8.1.4) by using another type of test function, i.e., $u_n-\eta\varphi_k(T_{2k}(u_n-v_0-T_h(u_n-v_0)+T_k(u_n)-T_k(u)))$ (see (8.3.10)).
It would be interesting at this stage to refer the reader to our previous work [4] in which, we studied the same problem but under some more restrictive conditions. We refer also the [74], where the author solved an analogous problem in the case of a Sobolev space, where the obstacle function verified $\psi^+\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

8.2. Main Results

Given an obstacle function $\psi:\Omega\to\overline{I\!R}$, we consider

$$K_\psi=\{u\in W_0^{1,p}(\Omega,w);\ u\geq\psi\ \ a.e.\ \text{in}\ \ \Omega\}. \tag{8.2.1}$$

Let A be the nonlinear operator from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ defined by

$$Au=-\text{div}(a(x,u,\nabla u)),$$

where $a:\Omega\times I\!R\times I\!R^N\to I\!R^N$ is a Carathéodory function satisfying the following assumptions:
(H_2') For $i=1,...,N$

$$|a_i(x,s,\xi)|\leq w_i^{\frac{1}{p}}(x)[k(x)+\sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}}+\sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{8.2.2}$$

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0\ \ \text{for all}\ \ \xi\neq\eta\in I\!R^N, \tag{8.2.3}$$

there exists $\delta(x)$ in $L^1(\Omega)$ and a strictly positive constant α such that, for some fixed element v_0 in $K_\psi\cap L^\infty(\Omega)$

$$a(x,s,\zeta)(\zeta-\nabla v_0)\geq\alpha\sum_{i=1}^{N} w_i(x)|\zeta_i|^p-\delta(x) \tag{8.2.4}$$

for a.e. x in $\Omega, s\in I\!R$ and all $\zeta\in I\!R^N$, where $k(x)$ is a positive function in $L^{p'}(\Omega)$.
Moreover, let $g(x,s,\xi)$ a Carathéodory function satisfying the conditions (H_3) of Chapter VII. ,

$$g(x,s,\xi).s\geq 0 \tag{8.2.5}$$

and

$$|g(x,s,\xi)| \le b(|s|)\left(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x)\right), \tag{8.2.6}$$

where $b : I\!R^+ \to I\!R^+$ is a positive increasing function and $c(x)$ is a positive function in $L^1(\Omega)$.

Our main result is the following.

Theorem 8.2.1. *Assume that* (H_1), (H_2') *and* (H_3) *hold and* $f \in L^1(\Omega)$. *Then there exists at least one solution of the following unilateral problem,*

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega, w), u \ge \psi \text{ a.e. in } \Omega, g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad \displaystyle\le \int_\Omega f T_k(u-v)\,dx, \\ \forall \;\; v \in K_\psi \cap L^\infty(\Omega), \;\forall k > 0. \end{cases} \tag{8.2.7}$$

Remark 8.2.1. *We remark that the statement of the previous Theorem does not exists in the case of Sobolev space. But, some existence results in this sense have been proved under the regularity assumption* $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, *(see [74]).*

Remark 8.2.2. *We obtain the same result if we assume only that the sign condition (8.2.5) is verified at infinity, or if the data is the form* $f - \mathrm{div}F$, *with* $f \in L^1(\Omega)$ *and* $F \in \prod L^{p'}(\Omega, w_i^{1-p'})$.

8.3. Proof of Theorem 8.2.1

STEP 1 : Approximate problems

Let us define

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}\theta_n(x)$$

where $\theta_n(x) = nT_{1/n}(\sigma^{1/q}(x))$.
Let us consider the approximate problems:

$$\begin{cases} u_n \in K_\psi, \\ \displaystyle\langle Au_n, u_n - v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \le \int_\Omega f_n(u_n - v)\,dx \\ \forall v \in K_\psi, \end{cases} \tag{8.3.1}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$.
Note that, $g_n(x,s,\xi)$ satisfies the following conditions:

$$g_n(x,s,\xi)s \ge 0, \;\; |g_n(x,s,\xi)| \le |g(x,s,\xi)| \;\text{ and }\; |g_n(x,s,\xi)| \le n.$$

We define the operator $G_n \; : W_0^{1,p}(\Omega, w) \longrightarrow W^{-1,p'}(\Omega, w^*)$ by,

$$\langle G_n u, v\rangle = \int_\Omega g_n(x, u, \nabla u) v \, dx$$

and

$$\langle Au, v\rangle = \int_\Omega a(x, u, \nabla u)\nabla v \, dx.$$

Thanks to Hölder's inequality, we have for all $u \in W_0^{1,p}(\Omega, w)$ and all $v \in W_0^{1,p}(\Omega, w)$,

$$\begin{aligned} \left|\int_\Omega g_n(x, u, \nabla u) v \; dx\right| &\leq \left(\int_\Omega |g_n(x, u, \nabla u)|^{q'} \sigma^{-\frac{q'}{q}} \; dx\right)^{\frac{1}{q'}} \left(\int_\Omega |v|^q \sigma \; dx\right)^{\frac{1}{q}} \\ &\leq n^2 \left(\int_\Omega \sigma^{q'/q} \sigma^{-q'/q} \; dx\right)^{\frac{1}{q'}} \|v\|_{q,\sigma} \\ &\leq C_n \|v\|. \end{aligned} \tag{8.3.2}$$

Lemma 8.3.1. *The operator $B_n = A + G_n$ from K_ψ into $W^{-1,p'}(\Omega, w^*)$ is pseudomonotone. Moreover, B_n is coercive in the following sense:*

$$\frac{< B_n v, v - v_0 >}{\|v\|} \longrightarrow +\infty \quad \textit{if} \;\; \|v\| \longrightarrow +\infty, v \in K_\psi.$$

This Lemma will be proved below.
In view of Lemma 8.3.1, the problem (8.3.1) has a solution by the classical result (cf. Theorem 8.2 Chapter 2 of [92]).

STEP 2 : A priori estimates

Let $k \geq \|v_0\|_\infty$ and let $\varphi_k(s) = s e^{\gamma s^2}$, where $\gamma = (\frac{b(k)}{\alpha})^2$.
It is well known that

$$\varphi_k'(s) - \frac{b(k)}{\alpha} |\varphi_k(s)| \geq \frac{1}{2}, \quad \forall s \in I\!R. \tag{8.3.3}$$

Taking $u_n - \eta \varphi_k(T_l(u_n - v_0))$ ($\eta = e^{-\gamma l^2}$) as test function in (8.3.1), where $l = k + \|v_0\|_\infty$, we obtain

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \nabla T_l(u_n - v_0) \varphi_k'(T_l(u_n - v_0)) \; dx + \int_\Omega g_n(x, u_n, \nabla u_n) \varphi_k(T_l(u_n - v_0)) \; dx \\ \leq \int_\Omega f_n \varphi_k(T_l(u_n - v_0)) \; dx. \end{aligned}$$

Since $g_n(x, u_n, \nabla u_n)\varphi_k(T_l(u_n - v_0)) \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, then

$$\begin{aligned} &\int_{\{|u_n - v_0| \leq l\}} a(x, u_n, \nabla u_n) \nabla(u_n - v_0) \varphi_k'(T_l(u_n - v_0)) \; dx \\ &\quad \leq \int_{\{|u_n| \leq k\}} |g_n(x, u_n, \nabla u_n)| |\varphi_k(T_l(u_n - v_0))| \; dx + \int_\Omega f_n \varphi_k(T_l(u_n - v_0)) \; dx. \end{aligned}$$

By using (8.2.4) and (8.2.6), we have

$$\alpha\int_{\{|u_n-v_0|\le l\}}\sum_{i=1}^{N}w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\varphi_k'(T_l(u_n-v_0))\,dx$$
$$\le b(|k|)\int_\Omega\left(c(x)+\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\right)|\varphi_k(T_l(u_n-v_0))|\,dx$$
$$+\int_\Omega\delta(x)\varphi_k'(T_l(u_n-v_0))\,dx+\int_\Omega f_n\varphi_k(T_l(u_n-v_0))\,dx.$$

Since $\{x\in\Omega,|u_n(x)|\le k\}\subseteq\{x\in\Omega:|u_n-v_0|\le l\}$ and the fact that $h,\delta\in L^1(\Omega)$, further f_n is bounded in $L^1(\Omega)$, then

$$\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\varphi_k'(T_l(u_n-v_0))\,dx$$
$$\le\tfrac{b(k)}{\alpha}\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p|\varphi_k(T_l(u_n-v_0))|\,dx+C_k$$

where C_k is a positive constant depending on k. This implies that

$$\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\left[\varphi_k'(T_l(u_n-v_0))-\frac{b(k)}{\alpha}|\varphi_k(T_l(u_n-v_0))|\right]dx\le C_k.$$

By using (8.3.3), we deduce

$$\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\,dx\le 2C_k. \tag{8.3.4}$$

STEP 3 : Convergence in measure of u_n

Let $k_0\ge\|v_0\|_\infty$ and $k>k_0$. Taking $v=u_n-T_k(u_n-v_0)$ as a test function in (8.3.1), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v_0)\,dx$$
$$\le\int_\Omega f_nT_k(u_n-v_0)\,dx. \tag{8.3.5}$$

Since $g_n(x,u_n,\nabla u_n)T_k(u_n-v_0)\ge 0$ on the subset $\{x\in\Omega,|u_n(x)|>k_0\}$, then (8.3.5) implies

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx\le k\int_{\{|u_n|\le k_0\}}|g_n(x,u_n,\nabla u_n)|\,dx+k\|f\|_{L^1(\Omega)},$$

which gives by using (8.2.6)

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx$$
$$\le kb(k_0)\left[\int_\Omega|c(x)|\,dx+\int_\Omega\sum_{i=1}^{N}w_i(x)|\frac{\partial T_{k_0}(u_n)}{\partial x_i}|^p\,dx\right]+kC. \tag{8.3.6}$$

Combining (8.3.4) and (8.3.6), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \le k[C_{k_0}+C].$$

Thanks to (8.2.4), we obtain

$$\int_{\{|u_n-v_0|\le k\}}\sum_{i=1}^N w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx \le kC_1$$

where C_1 is independent of k. Since k is arbitrary, we have

$$\int_{\{|u_n|\le k\}}\sum_{i=1}^N w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx \le \int_{\{|u_n-v_0|\le k+\|v_0\|_\infty\}}\sum_{i=1}^N w_i(x)|\frac{\partial u_n}{\partial x_i}|^p\,dx \le kC_2$$

i.e.,

$$\int_\Omega\sum_{i=1}^N w_i(x)|\frac{\partial T_k(u_n)}{\partial x_i}|^p\,dx \le kC_2. \tag{8.3.7}$$

Reasoning as in Chapter VII we can extract a subsequence still denoted by u_n, which converges almost everywhere to some function u, such that

$$\begin{array}{lll} T_k(u_n)\rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega,w), \\ T_k(u_n)\to T_k(u) & \text{strongly in } L^q(\Omega,\sigma) \text{ and } a.e. \text{ in } \Omega. \end{array} \tag{8.3.8}$$

This yields, by using (8.2.2), for all $k>0$ the existence of a function $h_k\in\prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'})$, such that

$$a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup h_k \text{ weakly in } \prod_{i=1}^N L^{p'}(\Omega,w_i^{1-p'}). \tag{8.3.9}$$

STEP 4 : Strong convergence of truncation.

We fix $k>\|v_0\|_\infty$, and let $w_{n,h}=T_{2k}(u_n-v_0-T_h(u_n-v_0)+T_k(u_n)-T_k(u))$ and $w_h=T_{2k}(u-v_0-T_h(u-v_0))$, with $h>2k$.
For $\eta=\exp(-4\gamma \mathrm{k}^2)$, we defined the following function as

$$v_{n,h}=u_n-\eta\varphi_k(w_{n,h}). \tag{8.3.10}$$

By taking $v_{n,h}$ as test functions in (8.3.1), we get

$$\langle A(u_n),\eta\varphi_k(w_{n,h})\rangle+\int_\Omega g_n(x,u_n,\nabla u_n)\eta\varphi_k(w_{n,h})\,dx \le \int_\Omega f_n\eta\varphi_k(w_{n,h})\,dx,$$

Since η is nonnegative, then

$$\langle A(u_n),\varphi_k(w_{n,h})\rangle+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx \le \int_\Omega f_n\varphi_k(w_{n,h})\,dx. \tag{8.3.11}$$

It follows that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\le\int_\Omega f_n\varphi_k(w_{n,h})\,dx. \tag{8.3.12}$$

Note that, $\nabla w_{n,h}=0$ on the set where $|u_n|>h+5k$; therefore, setting $m=5k+h$, and denoting by $\varepsilon_h^1(n),\varepsilon_h^2(n),...$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h, we get, by (8.3.12)

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx \le\int_\Omega f_n\varphi_k(w_{n,h})\,dx,$$

By the almost everywhere convergence of u_n, we have

$$\varphi_k(w_{n,h})\rightharpoonup\varphi_k(w_h)\quad\text{weakly}^*\ \text{ as }\ n\to+\infty\ \text{ in }\ L^\infty(\Omega). \tag{8.3.13}$$

Therefore,

$$\int_\Omega f_n\varphi_k(w_{n,h})\,dx\to\int_\Omega f\varphi_k(w_h)\,dx\ \text{ as }\ n\to+\infty. \tag{8.3.14}$$

On the set $\{x\in\Omega,|u_n(x)|>k\}$, we have $g(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\ge0$. So by (8.3.12), (8.3.14)

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx+\int_{\{|u_n|\le k\}}g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx \le\int_\Omega f\varphi_k(w_h)\,dx+\varepsilon_h^1(n). \tag{8.3.15}$$

Splitting the first integral on the left hand side of (8.3.15) where $|u_n|\le k$ and $|u_n|>k$, we can write,

$$\begin{aligned}\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx &=\int_{\{|u_n|\le k\}}a(x,T_m(u_n),\nabla T_m(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx\\&+\int_{\{|u_n|>k}a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx.\end{aligned} \tag{8.3.16}$$

The first term of the right hand side of the last equality can be written as

$$\begin{aligned}\int_{\{|u_n|\le k\}}a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx &\ge\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\,dx\\&-\varphi_k'(2k)\int_{\{|u_n|>k\}}\sum_{i=1}^N|a_i(x,T_k(u_n),0)||\frac{\partial T_k(u)}{\partial x_i}|\,dx.\end{aligned} \tag{8.3.17}$$

Recalling that, for $i=1,...,N$ $|a_i(x,T_k(u_n),0)|\chi_{\{|u_n|>k\}}$ converges to $|a_i(x,T_k(u),0)|\chi_{\{|u|>k\}}$ strongly in $L^{p'}(\Omega,w_i^{1-p'})$, moreover, since $|\frac{\partial T_k(u)}{\partial x_i}|\in L^p(\Omega,w_i)$, then

$$-\varphi_k'(2k)\int_{\{|u_n|>k\}}\sum_{i=1}^{N}|a_i(x,T_k(u_n),0)||\frac{\partial T_k(u)}{\partial x_i}|\;dx=\varepsilon_h^2(n).$$

For the second term of the right hand side of (8.3.16) we can write, using (8.2.4)

$$\begin{aligned}\int_{\{|u_n|>k\}} & a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\;dx\\ &\geq -\varphi_k'(2k)\int_{\{|u_n|>k\}}\sum_{i=1}^{N}|a_i(x,T_m(u_n),\nabla T_m(u_n))||\frac{\partial T_k(u)}{\partial x_i}|\;dx\\ &-\varphi'(2k)\int_{\{|u_n-v_0|>h\}}\delta(x)\;dx.\end{aligned}\tag{8.3.18}$$

Since for $i=1,....,N$ $(a_i(x,T_m(u_n),\nabla T_m(u_n)))_n$ is bounded in $L^{p'}(\Omega,w_i^{1-p'})$, it follows that the first term in the right hand side of the previous inequality tends to zero for every h fixed.
On the other hand, since $\delta\in L^1(\Omega)$, it is easy to see that

$$-\varphi_k'(2k)\int_{\{|u_n-v_0|>h\}}\delta(x)\;dx=-\varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\;dx+\epsilon_h^3(n).\tag{8.3.19}$$

Combining (8.3.16), .., (8.3.19), we deduce

$$\begin{aligned}\int_{\Omega} & a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\;dx\\ &\geq\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\;dx\\ &-\varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\;dx+\epsilon_h^4(n).\end{aligned}\tag{8.3.20}$$

This implies that

$$\begin{aligned}\int_{\Omega} & a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\;dx\\ &\geq\int_{\Omega}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\;dx\\ &+\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]\varphi_k'(w_{n,h})\;dx\\ &-\varphi_k'(2k)\int_{\{|u-v_0|>h\}}\delta(x)\;dx+\epsilon_h^4(n).\end{aligned}\tag{8.3.21}$$

We claim that

$$\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u))]\varphi_k'(w_{n,h})\;dx=\epsilon_h^5(n).\tag{8.3.22}$$

Indeed, since $\{x\in\Omega,|u_n(x)|\leq k\}\subseteq\{x\in\Omega:|u_n-v_0|\leq h\}$, we have

$$\begin{aligned}&\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u))]\varphi_k'(w_{n,h})\;dx\\ &=\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)\varphi_k'(T_k(u_n)-T_k(u))\;dx\\ &-\int_{\Omega}a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)\varphi_k'(w_{n,h})\;dx.\end{aligned}\tag{8.3.23}$$

By the continuity of the Nemitskii operator (see [68]), we have for all $i = 1, ..., N$,

$$a_i(x, T_k(u_n), \nabla T_k(u))\varphi'(T_k(u_n) - T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))\varphi'(0)$$

and

$$a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$$

strongly in $L^{p'}(\Omega, w_i^{1-p'})$, while $\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega, w_i)$, and $\frac{\partial(T_k(u))}{\partial x_i}\varphi'(w_{n,h}) \to \frac{\partial(T_k(u))}{\partial x_i}\varphi'(0)$ strongly in $L^p(\Omega, w_i)$.
This implies

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla T_k(u_n)\varphi_k'(T_k(u_n) - T_k(u))\, dx \\ = \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\varphi'(0)\, dx. \end{aligned} \tag{8.3.24}$$

and

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla T_k(u)\varphi_k'(w_{n,h})\, dx \\ = \int_\Omega a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\varphi'(0)\, dx. \end{aligned} \tag{8.3.25}$$

Combining (8.3.24) and (8.3.25), we deduce (8.3.22).
So that (8.3.21) and (8.3.22) yield

$$\begin{aligned} &\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(w_{n,h})\, dx \\ &\qquad \geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\ &\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)]\varphi'(w_{n,h})\, dx \\ &\qquad -\varphi_k'(2k)\int_{\{|u-v_0|>h\}} \delta(x)\, dx + \epsilon_h^6(n). \end{aligned} \tag{8.3.26}$$

We now, turn to the second term of the left hand side of (8.3.15), using (8.2.6), we have

$$\begin{aligned} &\left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,h})\, dx\right| \\ &\qquad \leq b(k)\int_\Omega \left(c(x) + \sum_{i=1}^N w_i |\frac{\partial T_k(u_n)}{\partial x_i}|^p\right) |\varphi_k(w_{n,h})|\, dx \\ &\qquad \leq b(k)\int_\Omega c(x)|\varphi_k(w_{n,h})|\, dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,h})| \\ &\qquad +\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,h})|\, dx \\ &\qquad -\tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla v_0|\varphi_k(w_{n,h})|\, dx, \end{aligned}$$

and hence (8.3.9) and (8.3.13) infer

$$\begin{aligned} &\left|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(w_{n,h})\, dx\right| \\ &\qquad \leq \tfrac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_{n,h})|\, dx \\ &\qquad +b(k)\int_\Omega c(x)|\varphi_k(w_h)|\, dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_h)|\, dx \\ &\qquad -\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\, dx + \epsilon_h^7(n). \end{aligned} \tag{8.3.27}$$

The first term of the right hand side of the last inequality can be written as

$$\begin{aligned}\tfrac{b(k)}{\alpha}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_{n,h})|\,dx\\ &+\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u))]|\varphi_k(w_{n,h})|\,dx\\ &+\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi_k(w_{n,h})|\,dx\end{aligned}\tag{8.3.28}$$

By Lebesgue's Theorem, we deduce that

$$\nabla T_k(u)|\varphi_k(w_{n,h})|\to\nabla T_k(u)|\varphi_k(T_{2k}(u-v_0-T_h(u-v_0)))|=0\ \text{ strongly in }\prod_{i=1}^N L^p(\Omega,w_i).$$

Which and using (8.3.9) implies that the third term of (8.3.28) tends to 0 as $n\to\infty$. On the other side reasoning as in (8.3.22), the second term of (8.3.28) tends to 0 as $n\to\infty$.
From (8.3.27) and (8.3.28), we obtain

$$\begin{aligned}\left|\int_{\{|u_n|\le k\}}g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\right|&\\ \le\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_{n,h})|\,dx\\ +b(k)&\int_\Omega c(x)|\varphi_k(w_h)|\,dx+\tfrac{b(k)}{\alpha}\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx\\ -\tfrac{b(k)}{\alpha}&\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+\epsilon_h^8(n).\end{aligned}\tag{8.3.29}$$

Combining (8.3.15), (8.3.26) and (8.3.29), we obtain

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)](\varphi_k'(w_{n,h})-\tfrac{b(k)}{\alpha}|\varphi_k(w_{n,h})|)\,dx\\ &\le b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx+c_k\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx\\ &-\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+\int_\Omega f(x)\varphi_k(w_h)\,dx+\epsilon_h^9(n).\end{aligned}\tag{8.3.30}$$

Then (8.3.3) implies

$$\begin{aligned}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\ &\le 2b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx+2c_k\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx\\ &-2\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+2\int_\Omega f(x)\varphi_k(w_h)\,dx+\epsilon_h^{10}(n).\end{aligned}\tag{8.3.31}$$

Hence, passing to the limit over n, we get

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega&[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\&\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\&\le 2b(k)\int_\Omega c(x)|\varphi_k(w_h)|\,dx+2c_k\int_\Omega\delta(x)|\varphi_k(w_h)|\,dx\\&-2\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx+2\int_\Omega f(x)\varphi_k(w_h)\,dx\end{aligned}\tag{8.3.32}$$

Now, since $c(x),\delta(x),f(x)$ and $h_k\nabla v_0$ belongs to $L^1(\Omega)$, by Lebesgue's dominated convergence, all the terms of the right hand side of the last inequality tend to 0 as $h\to+\infty$.
This implies that

$$\lim_{n\to\infty}\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx=0.$$

Finally, Lemma 6.4.1 of Chapter VII implies that

$$T_k(u_n)\to T_k(u)\ \text{ strongly in }\ W_0^{1,p}(\Omega,w)\ \ \forall k>0.\tag{8.3.33}$$

Since k arbitrary, we have for a subsequence

$$\nabla u_n\to\nabla u\ \text{ a.e. in }\ \Omega.\tag{8.3.34}$$

Which yields

$$\begin{cases}a(x,u_n,\nabla u_n)\to a(x,u,\nabla u)\ \text{ a.e. in }\ \Omega\\ g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\ \text{ a.e. in }\ \Omega.\end{cases}\tag{8.3.35}$$

STEP 5. Equi-integrability of the nonlinearities.

We need to prove that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\ \text{ strongly in }\ L^1(\Omega);\tag{8.3.36}$$

in particular it is enough to prove the equi-integrable of $g_n(x,u_n,\nabla u_n)$. To this purpose, we take $u_n-T_1(u_n-v_0-T_h(u_n-v_0))$ (with h large enough) as test function in (8.3.1); we obtain

$$\int_{\{|u_n-v_0|>h+1\}}|g_n(x,u_n,\nabla u_n)|\,dx\le\int_{\{|u_n-v_0|>h\}}(|f_n|+\delta(x))\,dx.$$

Let $\varepsilon>0$, then there exists $h(\varepsilon)\ge1$ such that

$$\int_{\{|u_n-v_0|>h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|\,dx<\varepsilon/2.\tag{8.3.37}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g_n(x,u_n,\nabla u_n)|\,dx\ &\le\ \int_E b(h(\varepsilon)+\|v_0\|_\infty)\left(c(x)+\sum_{i=1}^N w_i\left|\tfrac{\partial T_{h(\varepsilon)+\|v_0\|_\infty}(u_n)}{\partial x_i}\right|^p\right)\,dx\\&+\int_{\{|u_n-v_0|>h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|\,dx.\end{aligned}\tag{8.3.38}$$

In view of (8.3.33) there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b(h(\varepsilon) + \|v_0\|_\infty)\left(c(x) + \sum_{i=1}^N w_i |\frac{\partial T_{h(\varepsilon)+\|v_0\|_\infty}(u_n)}{\partial x_i}|^p\right)\, dx < \varepsilon/2 \tag{8.3.39}$$

for all E such that meas $E < \eta(\varepsilon)$.
Finally, combining (8.3.37), (8.3.38) and (8.3.39), one easily has

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx < \varepsilon \quad \text{for all } E \text{ such that meas } E < \eta(\varepsilon),$$

which implies (8.3.36).

STEP 6. Passing to the limit

Let $v \in K_\psi \cap L^\infty(\Omega)$, we take $u_n - T_k(u_n - v)$ as test function in (8.3.1), we can write

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)T_k(u_n - v)\, dx \\ \leq \int_\Omega f_n T_k(u_n - v)\, dx.\end{aligned} \tag{8.3.40}$$

This implies

$$\begin{aligned}\int_{\{|u_n - v| \leq k\}} & a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\, dx \\ & + \int_{\{|u_n - v| \leq k\}} a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla(v_0 - v)\, dx \\ & + \int_\Omega g_n(x, u_n, \nabla u_n)T_k(u_n - v)\, dx \leq \int_\Omega f_n T_k(u_n - v)\, dx.\end{aligned} \tag{8.3.41}$$

By Fatou's Lemma and the fact that

$$a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))$$

weakly in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ one can easily see that

$$\begin{aligned}\int_{\{|u - v| \leq k\}} & a(x, u, \nabla u)\nabla(u - v_0)\, dx \\ & + \int_{\{|u - v| \leq k\}} a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))\nabla(v_0 - v)\, dx \\ & + \int_\Omega g(x, u, \nabla u)T_k(u - v)\, dx \leq \int_\Omega f T_k(u - v)\, dx.\end{aligned} \tag{8.3.42}$$

Hence

$$\begin{aligned}\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega g(x, u, \nabla u)T_k(u - v)\, dx \\ \leq \int_\Omega f T_k(u - v)\, dx.\end{aligned} \tag{8.3.43}$$

This proves Theorem 7.3.1.

Proof of Lemma 8.3.1

From Hölder's inequality, the growth condition (8.2.2) we can show that A is bounded, and by using (8.3.2), we have B_n bounded. The coercivity follows from (8.2.4), (8.2.5) and (8.3.2). It remains to show that B_n is pseudo-monotone.
Let be a sequence $(u_k)_k \in K_\psi$, such that

$$u_k \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p}(\Omega, w),$$

$$\limsup_{k\to+\infty} \langle B_n u_k, u_k - u\rangle \le 0. \tag{8.3.44}$$

Let $v \in W_0^{1,p}(\Omega, w)$; we will prove that

$$\liminf_{k\to+\infty} \langle B_n u_k, u_k - v\rangle \ge \langle B_n u, u - v\rangle.$$

Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p}(\Omega, w)$, we deduce that $(a(x, u_k, \nabla u_k))_k$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$, then there exists a function $h \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup h \quad \text{weakly in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}),$$

similarly, it is easy to see that $(g_n(x, u_k, \nabla u_k))_k$ is bounded in $L^{q'}(\Omega, \sigma^{1-q'})$, then there exists a function $\rho_n \in L^{q'}(\Omega, \sigma^{1-q'})$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup \rho_n \quad \text{weakly in} \quad Ł^{q'}(\Omega, \sigma^{1-q'}).$$

It is clear that

$$\begin{aligned}
\liminf_{k\to+\infty} \langle B_n u_k, u_k - v\rangle &= \liminf_{k\to+\infty} \langle A u_k, u_k\rangle - \int_\Omega h \nabla v \, dx + \int_\Omega \rho_n (u - v) \, dx \\
&= \liminf_{k\to+\infty} \int_\Omega a(x, u_k, \nabla u_k) \nabla u_k \, dx - \int_\Omega h \nabla v \, dx + \int_\Omega \rho_n (u - v) \, dx.
\end{aligned} \tag{8.3.45}$$

On the other hand, By condition (8.2.3), we have

$$\int_\Omega (a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u))(\nabla u_k - \nabla u) \, dx \ge 0$$

which implies that

$$\begin{aligned}
\int_\Omega a(x, u_k, \nabla u_k) \nabla u_k \, dx \ \ge\ & -\int_\Omega a(x, u_k, \nabla u) \nabla u \, dx + \int_\Omega a(x, u_k, \nabla u_k) \nabla u \, dx \\
& + \int_\Omega a(x, u_k, \nabla u) \nabla u_k \, dx,
\end{aligned}$$

hence

$$\liminf_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k) \nabla u_k \, dx \ge \int_\Omega h \nabla u \, dx. \tag{8.3.46}$$

Combining (8.3.45) and (8.3.46), we get

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle \geq -\int_\Omega h\nabla(u-v)\,dx + \int_\Omega \rho_n(u-v)\,dx. \tag{8.3.47}$$

Now, since v is arbitrary and $\lim_{k\to+\infty}\langle G_n u_k, u_k - u\rangle = 0$, we have by using (8.3.44) and (8.3.47)

$$\lim_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla(u_k-u)\,dx = 0$$

we deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))\nabla(u_k-u)\,dx = 0.$$

In view of Lemma 6.4.1 of Chapter VII, we have $\nabla u_k \to \nabla u$ a.e. in Ω,
which with (8.3.47) yields

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle \geq \langle B_n u, u-v\rangle.$$

This completes the proof of the Lemma.

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